APPENDIX

In order to derive the results given in Proposition 1 we first introduce, in matrix notation,

$$w = s^2 - \sigma^2 = \frac{1}{n-1}\hat{u}'\hat{u} - \sigma^2 = O_p(\frac{1}{\sqrt{n}}) \quad \text{(i)}$$

$$\bar{u} = \frac{e'u}{n} = O_p(\frac{1}{\sqrt{n}}) \quad \text{(ii)}$$

where u is an n x 1 vector of u_i satisfying (2.2), e is an n x 1 vector of unit elements and

$$M = I - \frac{ee'}{n} \quad \text{(iii)}$$

is an n x n idempotent matrix with tr (M) = n - 1. Then from (2.1) and (2.4)

$$\hat{\theta} = \frac{s^2}{\bar{y}^2} = \frac{(w+\sigma^2)}{\mu^2}[1+\frac{\bar{u}}{\mu}]^{-2} \quad \text{(iv)}$$

Expanding the right hand side we get

$$\hat{\theta} - \theta = f_{-1/2} + f_{-1} + f_{-3/2} \quad \text{(v)}$$

where, denoting f_{-r} as a term of $O_p(n^{-r})$

$$\begin{aligned} f_{-1/2} &= \frac{w}{\mu^2} - \frac{2\theta}{\mu}\bar{u} \\ f_{-1} &= -\frac{2}{\mu^3}w\bar{u} + \frac{3\theta}{\mu^2}\bar{u}^2 \\ f_{-3/2} &= \frac{3w\bar{u}^2}{\mu^4} - \frac{4\theta\bar{u}^3}{\mu^3} \end{aligned} \quad \text{(vi)}$$

Thus, the bias of $\hat{\theta}$ to O (n^{-1}) is given,

$$Bias\ (\hat{\theta}) = E\ (f_{-1/2}) + E\ (f_{-1}) \quad \text{(vii)}$$

Now we state the following results:

$$\sigma^{-2}\ E[u'Bu] = trB \tag{viii}$$

$$\sigma^{-3}\ E[u'Buu] = \gamma_1(I*B)e \tag{ix}$$

$$\begin{aligned}
\sigma^{-4}\ E[u'Buuu'] &= \gamma_2(I*B) + (trB)I + 2B \\
\sigma^{-5}\ E\{u'Auu'Bu.u] &= \gamma_3(I*A*B)e + \gamma_1[(trA+2A) \\
&\quad (I*B)e + (trB+2B)(I*A)e + 4(I*AB)e] \\
\sigma^{-6}\ E[u'Au.u'Bu.uu'] &= \gamma_4(I*A*B) + \gamma_2\ [tr(A*B)I + tr\,A(I*B) \\
&\quad +trB(I*A) + 4(I*AB) + 2A(I*B) + \\
&\quad +2B(I*A) + 2(I*B)A + 2(I*A)B] + \gamma_1^2[4(A*B) \\
&\quad +(I*A)ee'(I*B) + (I*B)ee'(I*A) \\
&\quad +2I*\{A(I*B)\}ee' + 2I*\{B(I*A)\}ee'] \\
&\quad +2(tr\ A)B + 4AB + 2(tr\ B)A + 4BA \\
&\quad +[2(tr\ AB) + tr(A)\ tr(B)]I.
\end{aligned}$$

where A and B are n x n symmetric matrixes with non-stochastic elements—see, e.g. Ullah et. al. (1984, p. 398).

Using (i), (ii), (viii), and (ix) it is easy to see that

$$\begin{aligned} Ef_{-1/2} &= 0 \qquad \text{(xi)} \\ Ef_{-1} &= \frac{1}{n}\theta^{3/2}(3\theta^{1/2} - 2\gamma_1) \end{aligned}$$

which, when substituted in (vii) gives result (2.6) of the paper.

For obtaining the result (2.7), we write from (v), upto $O(n^{-2})$,

$$E(\hat{\theta} - \theta)^2 = E\ f_{-1/2}^2 + Ef_{-1}^2 + 2Ef_{-1/2}f_{-3/2} + 2Ef_{-1/2}f_{-1}.$$

It is easy to verify that

$$\begin{aligned} Ef_{-1/2}^2 &= \theta^2\left(\frac{\gamma_2}{n} + \frac{2}{n-1}\right) - 4\theta^{5/2}\frac{\gamma_1}{n} + \frac{4\theta^3}{n} \\ Ef_{-1}^2 + 2Ef_{-1/2}\ f_{-3/2} &= \frac{\theta^3}{n^2}\left(10_{\gamma 2}\frac{n+3}{n-1} + \frac{20n}{n-1} - 96\theta^{1/2}\gamma_1 + 75\theta + 20\gamma_1^2\frac{(n+1)}{(n-1)}\right) \\ 2Ef_{-1/2}f_{-1} &= \frac{\theta^{5/2}}{n^2}\left(14\theta^{1/2}\gamma_2\left(\frac{n}{n-1}\right) - 4\gamma_3 - 16\gamma_1\frac{n}{n-1} - 12\theta\gamma_1\right) \end{aligned}$$

To prove the result in proposition three, it is necessary to expand the sample skewness coefficient as well as the coefficient of variation squared. Letting $v = \widehat{\mu} - \mu = \frac{1}{n}\sum \widehat{u}_i^3 - \mu$ and

$$\widehat{\gamma}_1 - \gamma_1 = g_{-1/2} + g_{-1} + g_{-3/2}$$

the mean squared error of $\widetilde{\theta}$ is

$$\begin{aligned} MSE\left(\widetilde{\theta}\right) &= MSE\left(\widehat{\theta}\right) + \frac{1}{n^2}\left\{9\theta^4 + 4\theta^3\gamma_1^2 - 12\gamma_1\theta^{7/2}\right\} \\ &\quad -\frac{6}{n}\theta E\left(\theta f_{-1/2} + \theta f_{-1} + 2\theta f_{-1/2}^2\right) - \frac{4}{n}\theta^{3/2}\gamma_1 E\left(f_{-1/2} + f_{-1}\right) \\ &\quad -\frac{6}{n}\theta^{1/2}\gamma_1 Ef_{-1/2} - \frac{4}{n}\theta^{3/2}Ef_{-1/2}g_{-1/2} + o\left(\frac{1}{n^2}\right) \end{aligned}$$

and

$$Ef_{-1/2}g_{-1/2} = E\left[\frac{w}{\mu^2} - \frac{2\theta\overline{u}}{\mu}\right]\left[\frac{v}{\sigma^3} - \frac{3w\mu_3}{2\sigma^5}\right]$$

which upto $O(n^{-1})$, gives

$$Ef_{-1/2}g_{-1/2} = \frac{\theta}{n}\gamma_3 + \frac{6\theta}{n}\gamma_1 - \frac{2\theta^{\frac{3}{2}}}{n}\gamma_2 - \frac{3\theta}{n}\gamma_1\gamma_2 - \frac{3\theta}{n-1}\gamma_1 + \frac{3\theta^{\frac{3}{2}}}{n}\gamma_1^2.$$

Proposition 3 follows from straightforward algebra.