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Applications of Vector Autoregressions in Their Scalar Autoregressive Component Form

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Abstract

The eigenvalue/eigenvector structure underlying a standard N-variable P -lag vector autoregression (VAR) may be transformed into a system of NP scalar AR1 processes, each with an eigenvalue as its coefficient. This perspective allows a VAR to be assessed, analyzed, and manipulated using the mathematical and statistical convenience of elementary AR1 processes. Illustrative empirical applications demonstrate the inherent benefits: (1) the persistence of a VAR's dynamics is interpreted from its AR1 processes; (2) closed-form VAR forecasts are obtained from AR1 forecasts; (3) equality or zero constraints on selected AR1 coefficients are tested and imposed for VAR parsimony; (4) a median-unbiased estimate of the largest AR1 coefficient is generated and imposed to produce a more persistent VAR; (5) a unit root for the largest AR1 coefficient is tested and imposed to produce a cointegrated VAR, which also produces an estimate of the associated cointegrating vector.

Keywords

vector autoregression, VAR, companion matrix, eigenvalues, eigenvectors

JEL Classification

C13, C32, C53

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Abstract

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1 Introduction

In this note, I show that a standard vector autoregression (VAR) may be transformed into a system of scalar first-order autoregression (AR1) processes that provides a beneficial perspective for assessing, analyzing, and manipulating VARs in empirical applications. In particular, as I highlight in the examples outlined further below, scalar processes are mathematically straightforward to work with and the statistical nature of an AR1 process is elementary and transparent compared to a VAR in its coefficient matrix form.

The basis for the transformation is decomposing the companion matrix for a VAR into its eigenvalue and eigenvector matrices where, as detailed in section 2, the latter has a form that is well-known in applied mathematics; e.g. see Wilkinson (1965). Section 3 then establishes that the eigensystem decomposition for an N-variable P-lag VAR creates a system of NP scalar AR1 processes, each with an eigenvalue as its coefficient. Additionally, I show that the data used to estimate a VAR may be transformed into the sum of NP components, where each component is a constant $N \times 1$ vector containing the unique parameters of the eigenvector multiplied by its associated AR1 process. An immediate consequence is that point forecasts from the VAR for any chosen horizon may be obtained in closed-form, simply as elementary AR1 forecasts applied to the N-vectors. Forecast Error Variances may also be obtained in closed form, i.e. without

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recursive calculations through intervening horizons, which extends to ergodic variances as the infinite limit.

In section 4, I present five empirical applications, all based on an initial three-variable VAR estimated with US macroeconomic data, to demonstrate what I will refer to as the eigensystem VAR (EVAR) component framework. My objective is to establish the breadth of topics to which the EVAR component may be beneficially applied within multivariate time series analysis, rather than providing an in-depth development of any single application; each would clearly require more comprehensive treatment than possible in a note.

The first application is decomposing the initial VAR and its data into its EVAR components to assess its underlying dynamics. The second application is constructing closed-form forecasts of the VAR variables based on its EVAR components.

The remaining applications demonstrate adjustments to the initial VAR suggested by the assessment of its EVAR components. Hence, the third application produces more parsimonious VARs by testing and imposing a zero constraint on the least persistent AR1 coefficient, and an equality constraint on near-equal coefficients. The fourth application generates and constrains the coefficient of the most persistent AR1 process to its median-unbiased value, akin to Andrews (1993), thereby producing a more persistent bias-corrected VAR. A constraint of 1 is tested and imposed in the fifth application, thereby producing a cointegrated VAR. The estimated cointegrating vector is the N-vector of unique eigenvector parameters associated with the unit eigenvalue imposed on the associated AR1 process.

Within the literature, I am only aware of two examples explicitly related to the EVAR component framework.[1] That is, Neumaier and Schneider (2001) presents an analogous eigensystem decomposition for assessing the dynamics of a VAR, like my first application, and Krippner (2024) develops a framework for specifying and estimating a VAR directly via its eigensystem parameters, which I employ in my last three applications. More generally, there is a well-established literature on forecasting with VARs, estimating VARs with coefficient constraints, VAR unit root testing, and estimating cointegrating vectors/VARs; e.g. see Hamilton (1994), Lütkepohl (2006), and Juselius (2018) for extensive references. Abadir, Hadri, and Tzavalis (1999), Lawford and Stamatogiannis (2009), and Engsted and Pederson (2014) are examples that investigate estimation biases in VARs. The related applications in this note offer new and appealing approaches to each of the preceding topics. Specifically, to the best of my knowledge, the existing literature contains no examples of closed-form Forecast Error Variance expressions for VARs, using direct eigenvalue constraints for VAR parsimony or to impose unit roots in a VAR, or making median-unbiased corrections to a VAR via its eigenvalues.

The remainder of this note follows the outline given above. Section 5 concludes and briefly discusses potential extensions of the illustrative applications and other aspects that the EVAR component framework could beneficially be applied to. Only essential proofs are contained within the main text; supporting and supplementary material is relegated to an online appendix.

2 VARs and their eigensystem

A mean-adjusted VAR, which is most convenient for the exposition in this note, conditional on the initial P observations of a given $N \times (P+T)$ dataset $\{\bar{y}_t\}_{1-P}^{T}$, may be expressed as:

$$\bar{y}_t = \beta_1 \bar{y}_{t-1} + \ldots + \beta_P \bar{y}_{t-P} + \varepsilon_t \tag{1}$$

where μ is the full-sample mean $\mu = \frac{1}{T+P} \sum_{t=1-P}^{T} y_t$, $\bar{y}_t = y_t - \mu$ is an $N \times 1$ vector of mean-adjusted data at time t, $\bar{y}_{t-p}$ is mean-adjusted data at time $t-p$ with p ranging from 1 to

[1] Calculating companion matrix eigenvalues is routine to assess the stability of an estimated VAR, but such checks are not intended to be part of a component framework.

P, β_p are $N \times N$ matrices of coefficients associated with $\bar{y}_{t-P}$, and ε_t is a vector of residuals with an assumed multivariate normal distribution $\varepsilon_t \sim N\left(0_{N\times 1}, \Omega_\varepsilon\right)$, with $0_{N\times 1}$ the $N \times 1$ vector of zeros and Ω_ε the $N \times N$ covariance matrix. For a VAR estimated with a constant, i.e. $y_t = \alpha + \tilde{\beta}_1 y_{t-1} + \ldots + \tilde{\beta}_P y_{t-P} + \tilde{\varepsilon}_t$, the mean $\tilde{\mu} = \left(I_N - \tilde{\beta}_1 - \ldots - \tilde{\beta}_P\right)^{-1} \alpha$ would be used to obtain $\bar{y}_t$, but otherwise the exposition remains the same.

The companion form of the mean-adjusted VAR is its equivalent re-expression as an NP-variable first-order VAR, i.e.:

$$\bar{Y}_t = B\bar{Y}_{t-1} + E_{Y,t} \tag{2}$$

where $\bar{Y}_t = \left[\bar{y}_t', \ldots, \bar{y}_{t-P+1}'\right]'$, $\bar{Y}_{t-1} = \left[\bar{y}_{t-1}', \ldots, \bar{y}_{t-P}'\right]'$, $E_{Y,t} = \left[\varepsilon_t', 0, \ldots, 0\right]'$ (all $NP \times 1$ vectors), and $E_{Y,t} \sim N\left(0_{NP\times 1}, \Omega_{E_Y}\right)$ with Ω_ε in the upper-left corner being the only non-zero part of the $NP \times NP$ covariance matrix Ω_{E_Y}. B is the $NP \times NP$ companion matrix:

$$B = \begin{bmatrix} \multicolumn{2}{c}{\beta_{N\times NP}} \\ I_{(N-1)P} & 0_{(N-1)P\times N} \end{bmatrix} \tag{3}$$

where $\beta = [\beta_1, \ldots, \beta_P]$, the $N \times NP$ matrix of VAR coefficients, and $I_{(N-1)P}$ and $0_{(N-1)P\times N}$ are the identity matrix and a zero matrix, respectively, with dimensions given in their subscripts.

The $N \times NP$ matrix $J = [I_N, 0_N, \ldots, 0_N]$ links the VAR and its companion form, i.e. $J\bar{Y}_t = J\left(B\bar{Y}_{t-1} + E_t\right)$ gives $\bar{y}_t = \beta\bar{Y}_{t-1} + \varepsilon_t$, where $\beta\bar{Y}_{t-1} = \beta_1\bar{y}_{t-1} + \ldots + \beta_P\bar{y}_{t-P}$, $J\Omega_{E_Y}J' = \Omega_\varepsilon$, and $J'\Omega_\varepsilon J = \Omega_{E_Y}$. Lütkepohl (2006) chapter 2 and Hamilton (1994) chapter 11 are standard references for the aspects outlined in this section so far.

The companion matrix B may be decomposed into its eigenvector matrix D and eigenvector matrix V (both $NP \times NP$), hence $B = VDV^{-1}$. The eigenvector matrix is $V = [V_1, \ldots, V_{NP}]$ where each V_k has the form given below:

$$V_k = \begin{bmatrix} S_k D_k^{P-1} \\ \vdots \\ S_k D_k \\ S_k \end{bmatrix} \text{ with } S_k = \begin{bmatrix} S_{1,k} \\ \vdots \\ S_{N-1,k} \\ 1 \end{bmatrix} \text{ or } V = \begin{bmatrix} SD^{P-1} \\ \vdots \\ SD \\ S \end{bmatrix} \text{ and } V_X = \begin{bmatrix} S \\ SD^{-1} \\ \vdots \\ SD^{P-1} \end{bmatrix} \tag{4}$$

with each S_k an $N \times 1$ vector, and I have set the last element to 1 for this note as the most convenient arbitrary eigenvector normalization. The first expression in equation 4 is from Wilkinson (1965) pp. 33-34, as also referenced in Neumaier and Schneider (2001), and applies in the case that all eigenvalues are distinct, hence $D = \text{diag}([D_1, \ldots, D_{NP}])$. The third expression is my generalized notation where $S = [S_1, \ldots, S_{NP}]$ and each block SD^{p-1} is an $N \times NP$ matrix. This form accommodates the distinct eigenvalue case, allows D to include Jordan blocks in the case of repeated eigenvalues, and also underlies the final expression $V_X = VD^{1-P}$ in equation 4 that is most convenient for the notation and derivations in this note. That is, V_X provides an equivalent companion matrix decomposition $B = V_X D V_X^{-1}$ (i.e. $B = VDV^{-1} = VD^{1-P}DVD^{P-1}V^{-1} = V_X D V_X^{-1}$), and $JV_X = S$. Section 2 in the online appendix shows that V and V_X must take the forms in equation 4.

3 VAR components and closed-form forecasts

This section first establishes the EVAR component structure in the case of distinct eigenvalues, then section 3.2 uses that structure to obtain closed-form expressions for VAR forecasts.

Note that D may in general include real values and complex conjugate pairs (CCPs), i.e. $\overline{D_{k+1}} = D_k$. Where required, I accommodate both cases by using the complex conjugate transpose (Hermitian) operator "†" ; i.e. for a generic vector or matrix C, $C^\dagger = \overline{C}'$ so $C_{i,j} = \overline{C_{j,i}}$, and $C^\dagger = C'$, the standard transpose, when C is real. Section 3 of the online appendix shows that

CCP EVAR components result from CCP eigenvalues, and Sekita, Kurita, and Otsu (1992) and Gu and Jiang (2005) are references for complex AR1 processes (which are AR1 processes with a complex coefficient and complex-valued data). In section 3.3, I briefly discuss an alternative to using complex AR1 processes, and also the generalization required when a VAR contains repeated eigenvalues.

3.1 VAR components

Proposition 1 (VAR transformation to AR1 processes) *If all eigenvalues are distinct, the companion form of an N-variable P-lag VAR may be expressed equivalently as a system of NP scalar AR1 processes.*

Proof.

$$\begin{aligned} \bar{Y}_t &= B\bar{Y}_{t-1} + E_{Y,t} \\ &= V_X D V_X^{-1} \bar{Y}_{t-1} + E_{Y,t} \\ V_X^{-1}\bar{Y}_t &= D V_X^{-1}\bar{Y}_{t-1} + V_X^{-1} E_{Y,t} \\ X_t &= D X_{t-1} + E_{X,t} \end{aligned} \tag{5}$$

where $X_t = V_X^{-1}\bar{Y}_t$, $X_{t-1} = V_X^{-1}\bar{Y}_{t-1}$, $E_{X,t} = V_X^{-1}E_{Y,t}$ (all $NP \times 1$ vectors), and $E_{X,t} \sim N\left(0_{NP\times 1}, \Omega_{E_X}\right)$ with $\Omega_{E_X} = V_X^{-1} J' \Omega_\varepsilon J \left(V_X^{-1}\right)^\dagger$. Given that D is diagonal in the case of distinct eigenvalues, each of the NP rows of equation 5 is:

$$X_{k,t} = D_k X_{k,t-1} + E_{Xk,t} \tag{6}$$

with $E_{Xk,t} \sim N\left(0, \Omega_{E_X,k,k}\right)$. ■

Proposition 2 (AR1 coefficient equivalence to VAR eigenvalues) *If all eigenvalues are distinct, each D_k is equivalent to an AR1 coefficient from the OLS regression of $X_{k,t}$ on $X_{k,t-1}$.*

Proof. A VAR conditional on the initial P observations of a given $N \times (P+T)$ dataset $\{\bar{y}_t\}_{1-P}^{T}$ has maximum log-likelihood coefficient estimates $\beta = \bar{y}\bar{Y}_L' \left(\bar{Y}_L \bar{Y}_L'\right)^{-1}$, where $\bar{y}$ is the $N \times T$ matrix of data $\bar{y}_t$ for all periods, and $\bar{Y}_L$ is the $NP \times T$ matrix of data $\bar{Y}_{t-1}$ for all periods; e.g. see Hamilton (1994) pp. 293-96 or my alterative proof in section 4 of the online appendix. Therefore, from equation 3:

$$\begin{aligned} B &= \begin{bmatrix} \bar{y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} \\ I_{NP-N} \quad 0_{[NP-N]\times N} \end{bmatrix} \\ &= \bar{Y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} \end{aligned} \tag{7}$$

where the second line uses the identity $\left[\bar{y}_{-1}', \ldots, \bar{y}_{-P+1}'\right]' \bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} = \left[I_{NP-N}, 0_{[NP-N]\times N}\right]$, and each $\bar{y}_{-p}$ is the $N \times T$ matrix of lagged data $\bar{Y}_{t-p}$. This identity is intuitive from the line-by-line OLS regression perspective for $\bar{Y}_t = B\bar{Y}_{t-1} + E_{Y,t}$, i.e. $y_{1,t-1} = [1, 0, \ldots, 0]\, Y_{t-1} + 0$, $y_{2,t-1} = [0, 1, 0, \ldots, 0]\, Y_{t-1} + 0$, etc., but section 4 of the online appendix includes a proof.

The $NP \times T$ matrices of X_t and X_{t-1} for all periods are respectively $X = V_X^{-1}\bar{Y}$ and $X_L = V_X^{-1}\bar{Y}_L$, and $DX_L X_L^\dagger = XX_L^\dagger$ is equivalent to $B = \bar{Y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$, i.e.:

$$\begin{aligned} DX_L X_L^\dagger &= XX_L^\dagger \\ DV_X^{-1}\bar{Y}_L\bar{Y}_L' \left(V_X^{-1}\right)^\dagger &= V_X^{-1}\bar{Y}\bar{Y}_L' \left(V_X^{-1}\right)^\dagger \\ DV_X^{-1}\bar{Y}_L\bar{Y}_L' &= V_X^{-1}\bar{Y}\bar{Y}_L' \\ V_X D V_X^{-1} &= \bar{Y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} \end{aligned} \tag{8}$$

where $X_L^\dagger = \left(V_X^{-1}\bar{Y}_L\right)^\dagger = \bar{Y}_L^\dagger \left(V_X^{-1}\right)^\dagger = \bar{Y}_L' \left(V_X^{-1}\right)^\dagger$ has been used in the second line. Equating the diagonal matrix elements of $DX_LX_L^\dagger$ and $XX_L^\dagger$ then gives the AR1 coefficients as OLS regressions, i.e.:

$$\begin{aligned} D_k X_{k,L} X_{k,L}^\dagger &= X_k X_{k,L}^\dagger \\ D_k &= X_k X_{k,L}^\dagger \left(X_{k,L} X_{k,L}^\dagger\right)^{-1} \end{aligned} \tag{9}$$

■

Proposition 3 ($\bar{y}_t$ is the sum of components $S_k X_{k,t}$) *If all eigenvalues are distinct, $\bar{y}_t$ is the sum of NP components, with each component being the vector S_k multiplied by the AR1 process for $X_{k,t}$.*

Proof. $\bar{y}_t = J\bar{Y}_t$ and $X_t = V_X^{-1}\bar{Y}_t$ gives $\bar{Y}_t = V_X X_t$, so $\bar{y}_t = JV_X X_t = SX_t$, which may be expressed in summation form as:

$$\bar{y}_t = \sum_{k=1}^{NP} S_k X_{k,t} \tag{10}$$

■

3.2 Closed-form VAR forecasts

Point forecasts from VARs are typically obtained recursively, i.e. $\mathbb{E}_t\left[\bar{y}_{t+1}\right] = \beta\bar{Y}_t$, $\mathbb{E}_t\left[\bar{y}_{t+2}\right] = \beta\bar{Y}_{t+1}$, etc., or in closed-form but with matrix powers, i.e. $\mathbb{E}_t\left[\bar{y}_{t+h}\right] = JB^h\bar{Y}_t$. Forecast Error Variances (FEVs) $\Omega_y(h)$ used to obtain confidence intervals around $\mathbb{E}_t\left[\bar{y}_{t+h}\right]$ are obtained recursively, i.e. $\Omega_y(1) = \Omega_\varepsilon$, $\Omega_y(2) = \Omega_y(1) + \Phi_1\Omega_\varepsilon\Phi_1'$, etc. with $\Phi_n = JB^nJ'$, or equivalently $\Omega_y(h) = \sum_{n=0}^{h-1}\Phi_n\Omega_\varepsilon\Phi_n'$. The ergodic variance is $\Omega_y(\infty) = \sum_{n=0}^{\infty}\Phi_n\Omega_\varepsilon\Phi_n'$. Lütkepohl (2006) section 2.2.2 is a reference for all of the preceding aspects.

The first two propositions below respectively show that the EVAR components simplify VAR point forecasts to a closed-form based on elementary AR1 forecasts, and also allows FEVs to be obtained in closed-form based on scalar powers of the eigenvalues. The infinite limit of the FEV is the closed-form the ergodic variance, and section 5 of the online appendix discusses this ergodic variance result within the wider context of multivariate ergodic variances.

Proposition 4 (Closed-form point forecasts for a VAR) *If all eigenvalues are distinct, $\mathbb{E}_t\left[y_{t+h}\right]$ is the sum of the NP components, with each being the vector S_k multiplied by forecasts of the AR1 process for X_t.*

Proof. Applying the expectations operator for horizon h to $\bar{y}_t = SX_t$ gives $\mathbb{E}_t\left[\bar{y}_{t+h}\right] = \mathbb{E}_t\left[SX_{t+h}\right] = S\mathbb{E}_t\left[X_{t+h}\right]$, and $X_t = DX_{t-1} + E_{X,t}$ gives $\mathbb{E}_t\left[X_{t+h}\right] = D^hX_t$. Therefore:

$$\mathbb{E}_t\left[\bar{y}_{t+h}\right] = SD^hX_t \tag{11}$$

D is diagonal in the case of distinct eigenvalues, so equation 11 may be expressed in summation form as:

$$\mathbb{E}_t\left[\bar{y}_{t+h}\right] = \sum_{k=1}^{NP} S_k D_k^h X_{k,t} \tag{12}$$

■

Proposition 5 (Closed-form FEVs for a VAR) *If all eigenvalues are distinct, $\Omega_y(h)$ may be obtained by calculating each of the (i,j) elements of $\Omega_X(h)$ as:*

$$[\Omega_X(h)]_{ij} = \Omega_{E_X,ij}\frac{1-\left(D_i\overline{D_j}\right)^h}{1-D_i\overline{D_j}} \tag{13}$$

where $\Omega_{E_X} = V_X^{-1}J'\Omega_\varepsilon J\left(V_X^{-1}\right)^\dagger$, an $NP\times NP$ Hermitian matrix, and then using the resulting $NP\times NP$ matrix $\Omega_X(h)$ in the expression:

$$\Omega_y(h) = S\Omega_X(h)S^\dagger \tag{14}$$

Proof. The FEV summation expression $\Omega_y(h) = \sum_{n=0}^{h-1}\Phi_n\Omega_\varepsilon\Phi_n'$ with $\Phi_n = JB^nJ'$ is $\Omega_y(h) = \sum_{n=0}^{h-1}JB^nJ'\Omega_\varepsilon J(B^n)'J'$. Using the decomposition $B = V_XDV_X^{-1}$:

$$\begin{aligned}\Omega_y(h) &= \sum_{n=0}^{h-1}JV_XD^nV_X^{-1}J'\Omega_\varepsilon J\left(V_XD^nV_X^{-1}\right)^\dagger J' \\ &= S\left[\sum_{n=0}^{h-1}D^n\Omega_{E_X}\left(D^\dagger\right)^n\right]S^\dagger\end{aligned} \tag{15}$$

and below I use $\Omega_X(h) = \sum_{n=0}^{h-1}D^n\Omega_{E_X}\left(D^\dagger\right)^n$ to denote the matrix defined by the summation in the square brackets.

For clarity, I use $ACA^\dagger$ as a generic expression for each matrix $D^n\Omega_{E_X}\left(D^\dagger\right)^n$, hence $A = D^n$ and $C = \Omega_{E_X}$, and then evaluate the elements using index notation for matrix products, i.e. $\left[ACA^\dagger\right]_{ij} = \sum_{k=1}^{NP}A_{ik}\left(\sum_{l=1}^{NP}C_{kl}A_{lj}^\dagger\right)$. First, given $A_{l\neq j}^\dagger = 0$, $\sum_{l=1}^{NP}C_{kl}A_{lj}^\dagger = C_{kj}A_{jj}^\dagger$, so $\left[ACA^\dagger\right]_{ij} = \sum_{k=1}^{NP}A_{ik}C_{kj}A_{jj}^\dagger$. Then, given $A_{i\neq k} = 0$, $\left[ACA^\dagger\right]_{ij} = A_{ii}C_{ij}A_{jj}^\dagger = C_{ij}A_{ii}\overline{A_{jj}}$. Therefore, each (i,j) element of $D^n\Omega_{E_X}\left(D^\dagger\right)^n$ is $\Omega_{E_X,ij}D_i^n\overline{D_j^n} = \Omega_{E_X,ij}\left(D_i\overline{D_j}\right)^n$, and each element of $[\Omega_X(h)]_{ij}$ is then:

$$\begin{aligned}[\Omega_X(h)]_{ij} &= \Omega_{E_X,ij}\sum_{n=0}^{h-1}\left(D_i\overline{D_j}\right)^n \\ &= \Omega_{E_X,ij}\frac{1-\left(D_i\overline{D_j}\right)^h}{1-D_i\overline{D_j}}\end{aligned} \tag{16}$$

where the sum of the geometric series $\left(D_i\overline{D_j}\right)^n$ has been replaced by its closed-form solution.

Each (i,j) element of $\Omega_X(h)$ may therefore be calculated using the closed-form solution given in equation 16, with $NP(NP+1)/2$ unique calculations required given the Hermitian symmetry. The resulting $\Omega_X(h)$ is used in equation 15 to obtain $\Omega_y(h) = S\Omega_X(h)S^\dagger$. ■

Proposition 6 (Closed-form ergodic variance for a VAR) *If all eigenvalues are distinct and less than 1 in magnitude, the ergodic variance $\Omega_y(\infty)$ may be obtained by calculating each of the (i,j) elements of $\Omega_X(\infty)$ as $[\Omega_X(\infty)]_{ij} = \Omega_{E_X,ij}/\left(1-D_i\overline{D_j}\right)$, and then using the resulting matrix $\Omega_X(\infty)$ in the expression $\Omega_y(\infty) = S\Omega_X(\infty)S^\dagger$.*

Proof. $\Omega_y(\infty) = S\left[\lim_{h\to\infty}\Omega_X(h)\right]S^\dagger = S\Omega_X(\infty)S^\dagger$, and each element of $\Omega_X(\infty)$ is obtained as $\lim_{h\to\infty}[\Omega_X(h)]_{ij} = \Omega_{E_X,ij}\left(\lim_{h\to\infty}\frac{1-\left(D_i\overline{D_j}\right)^h}{1-D_i\overline{D_j}}\right) = \Omega_{E_X,ij}\frac{1}{1-D_i\overline{D_j}}$. ■

3.3 CCP and repeated eigenvalues

An alternative to using CCP expressions in the case of CCP eigenvalues is to use real forms based on the real 2×2 AR2 companion matrix. This is detailed in section 6 of the online appendix but, as discussed further there, it is preferable to keep all AR1 processes separate for analysis and applications to fully exploit the inherent scalar nature associated with distinct eigenvalues. CCP results may then be combined into real forms as necessary, such as summing the CCP components in figure 1, or real trigonometric forms could also be used; again see section 6 of the online appendix.

Repeated eigenvalues will only occur in an estimated VAR if they are imposed, as in the third application in section 4 with a pair of repeated eigenvalues. This case requires a 2×2 Jordan block within D but, as detailed in section 7 of the online appendix, still obtains scalar closed-form solutions.

4 Empirical applications

This section contains five illustrative empirical applications to demonstrate the range of benefits that the EVAR component framework offers. Underlying these applications is an initial OLS VAR estimation on mean-adjusted end-quarter United States data for unemployment u_t, annual CPI inflation π_t, and the 3-month Treasury bill rate r_t (all from https://fred.stlouisfed.org). The sample is from 1948Q1 (the first period with complete data) to 2007Q3 (immediately prior to the onset of the Global Financial Crisis, simply to avoid the known change in the data-generating process for r_t due to subsequent periods with a lower bound constraint). The lag length is $P = 2$, as suggested by the Schwarz criterion.

Table 1 contains the VAR coefficient and covariance matrix estimates. Standard errors are omitted, here and elsewhere, to save space. The log-likelihood value for the initial VAR is $\mathcal{L}_0 = -668.18$.

Table 1: VAR coefficient and covariance matrix estimates

	coefficients						covariances		
	β_1			β_2			Ω_ε		
β_u	1.37	–0.02	0.03	–0.46	0.04	–0.00	0.12	–0.02	–0.13
β_π	–0.36	1.14	0.11	0.31	–0.25	–0.04	–0.02	0.69	0.21
β_r	–0.67	–0.07	0.73	0.67	0.15	0.17	–0.13	0.21	0.90

The first application decomposes the initial VAR into its EVAR components. Hence, table 2 contains the eigensystem parameters D_k and S_k that underlie the VAR in table 1. I obtain these from the VAR companion matrix B with the MatLab function $[V, D] = \text{eig}(B)$ and then re-normalize each V_k so its last element is 1. D_k are the diagonal elements of D and the vectors S_k are the last N rows of V_k. Below the eigensystem parameters are the diagonal elements of $\Omega_{E_X,k,k}$, i.e. the variance of the AR1 process $X_{k,t} = D_k X_{k,t-1} + E_{Xk,t}$ for each component.

The first component is very persistent, with an AR1 coefficient of 0.97 and a half-life, $-\log(2) / |D_k|$, of 24.30 quarters. The next three components are moderately persistent and the last two components are transitory. Components 3 and 4 are a CCP with an oscillation period of $2\pi / \left[\cos^{-1}(\text{Re}[D_3] / |D_3|)\right] = 34.00$ quarters and a half-life for the magnitude of 2.25 quarters. Component 6 has a two-quarter oscillation period. The ergodic variances for each AR1 process, $\Omega_{X,k,k}(\infty) = \Omega_{E_X,k,k} / \left(1 - |D_k|^2\right)$, show that the components make contributions to the VAR's dynamics in the same order as their persistence, i.e. component 1 dominates, components 2 to 4 make moderate contributions, and the contributions of components 5 and 6 are minor.

Table 2: VAR eigensystem parameters and EVAR component aspects

k	1	2	3	4	5	6
D_k	0.97	0.75	0.72+0.14i	0.72–0.14i	0.22	–0.15
$S_{k,u}$	0.52	–0.58	–1.76+0.28i	–1.76–0.28i	–0.21	–0.02
$S_{k,\pi}$	0.65	–0.53	1.18–0.98i	1.18+0.98i	–1.22	–0.14
$S_{k,r}$	1	1	1	1	1	1
$\Omega_{E_X,k,k}$	0.40	1.30	0.37	0.37	0.06	0.02
Half-life of $\lvert D_k \rvert$	24.30	2.14	2.25	2.25	0.46	0.36
$\Omega_{X,k,k}(\infty)$	7.19	2.97	0.80	0.80	0.06	0.02

The second application obtains closed-form forecasts for the VAR and its EVAR components. Hence, figure 1 plots the mean-adjusted data $\overline{y}_t$ and the point forecasts from the VAR out to an arbitrary horizon of 40 quarters, where T is the last quarter of the sample, i.e. 2007Q3. For just $\bar{r}_t$, to maintain clarity in the figure, I have also included the ± 1 standard deviation confidence interval and the ± 1 ergodic standard deviation.[2] These are respectively obtained as $\pm 1\sigma \text{FE}(h) = \sqrt{[\Omega_y(h)]_{3,3}}$ and $\pm 1\sigma \text{FE}(\infty) = \sqrt{[\Omega_y(\infty)]_{3,3}}$, where $\Omega_y(h)$ and $\Omega_y(\infty)$ are calculated using the closed-form expressions from section 3. $\Omega_y(\infty)$ for the VAR is:

$$\Omega_y(\infty) = \begin{bmatrix} 2.18 & 1.19 & 1.60 \\ 1.19 & 7.55 & 5.81 \\ 1.60 & 5.81 & 8.42 \end{bmatrix} ; \ [\Omega_y(\infty)]_1 = \begin{bmatrix} 1.92 & 2.40 & 3.72 \\ 2.40 & 3.01 & 4.65 \\ 3.72 & 4.65 & 7.19 \end{bmatrix} \quad (17)$$

and $[\Omega_y(\infty)]_1$ is discussed shortly below.

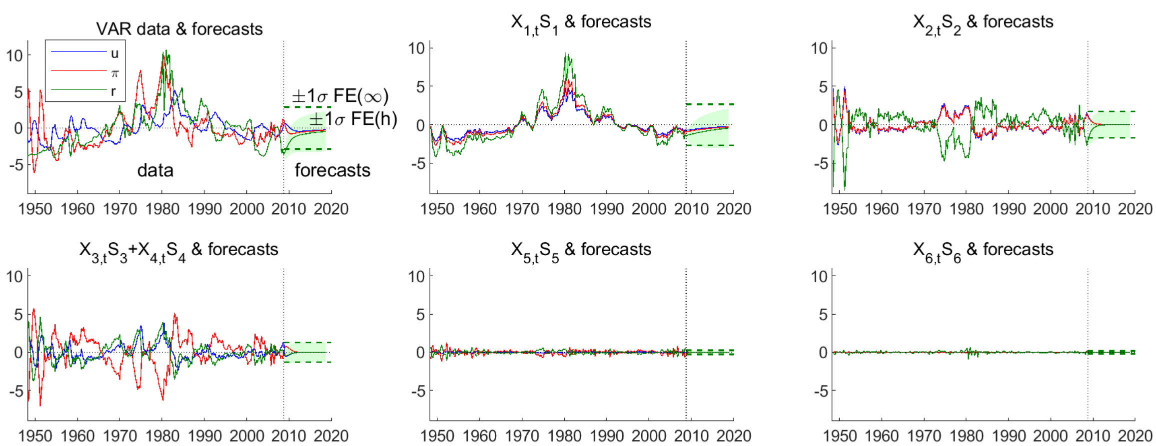

Figure 1: Panel 1 contains the VAR data and forecasts and, for r_t, the shaded ± 1 standard deviation confidence interval and the dashed ± 1 ergodic standard deviation. The remaining panels contain the same aspects for the EVAR components that underlie the VAR data and forecasts.

The remaining panels of figure 1 plot the components associated with each eigenvalue, using the same scale for comparability. For the real eigenvalues, each historical component is $S_k X_{k,t}$, the point forecasts are $S_k D_k^h X_{k,T}$. For r_t, $\pm 1\sigma \text{FE}(h) = \sqrt{[\Omega_y(h)]_{k,3,3}}$ where $[\Omega_y(h)]_k = \Omega_{X,k,k}(h) S_k S_k'$, and $\pm 1\sigma \text{FE}(\infty) = \sqrt{[\Omega_y(\infty)]_{k,3,3}}$ where $[\Omega_y(\infty)]_k = \Omega_{X,k,k}(\infty) S_k S_k'$. For example, from $[\Omega_y(\infty)]_1$ provided above, $[\Omega_y(\infty)]_{1,3,3} = 7.19$.

[2] The infinite limit of VAR point forecasts with all $|D_k| < 1$ is $\lim_{h\to\infty} \mathbb{E}_t [\overline{y}_{t+h}] = S [\lim_{h\to\infty} (D^h)] X_t = S [0_{N\times N}] X_t = 0_{N\times 1}$.

The variances $[\Omega_y(h)]_k$ and $[\Omega_y(\infty)]_k$ may also be viewed as contributions to $\Omega_y(h)$ and $\Omega_y(\infty)$ from the EVAR component k, and component 1 dominates in that regard. Section 8 of the online appendix contains the contributions of $[\Omega_y(\infty)]_k$ to $\Omega_y(\infty)$ for all components, including cumulative contributions that account for component covariances. Note that the scalar series $X_{k,t}$ (and $X_{3,t}+X_{4,t}$ below), and their forecasts and variances will be identical to the results for $\bar{r}_t$, due to my normalization of 1 for the last element of S_k.

The results for the CCP eigenvalues 3 and 4 are analogous to the real eigenvalue case, except they are treated in pairs to create real results. I have summed the CCP components and forecasts, i.e. $S_3X_{3,t}+S_4X_{4,t}$ and $S_3D_3^hX_{3,T}+S_4D_4^hX_{4,T}$. The variances are $[\Omega_y(h)]_{3\&4} = [S_3,S_4]\,\Omega_{X,3:4}(h)\,[S_3,S_4]^\dagger$ and $[\Omega_y(\infty)]_{3\&4} = [S_3,S_4]\,\Omega_{X,3:4}(\infty)\,[S_3,S_4]^\dagger$, where 3:4 denotes the 2×2 blocks from $\Omega_X(h)$ and $\Omega_X(\infty)$ associated with the eigenvalues (D_3, D_4); see section 6 of the online appendix for full details.

The third application is testing whether selected eigenvalues of the initial VAR may be restricted to achieve parsimony, and I test two candidates based on the assessment discussed earlier. First, given that D_6 is the smallest eigenvalue associated with the least important EVAR component, I test $D_6 = 0$. Second, given that the imaginary components for the (D_3, D_4) CCP are small then $D_3 \simeq D_4$, so I test $D_3 = D_4$ within a Jordan block. In both cases, I re-estimate the VAR with the given constraints using the method provided in Krippner (2024), which results in log-likelihood values of $\mathcal{L}_1 = -668.82$ and $\mathcal{L}_1 = -668.50$. The log-likelihood ratios $-2(\mathcal{L}_1-\mathcal{L}_0)$ are therefore 1.28 and 0.65 relative to the initial VAR, with probability values $\chi^2(1.28,1) = 0.742$ and $\chi^2(0.65,1) = 0.580$. Hence, constraining the VAR with either $D_6 = 0$ or $D_3 = D_4$ is not rejected at any standard level of significance. The coefficients and eigensystem for each of the constrained VAR estimations are contained respectively in sections 9 and 10 of the online appendix.

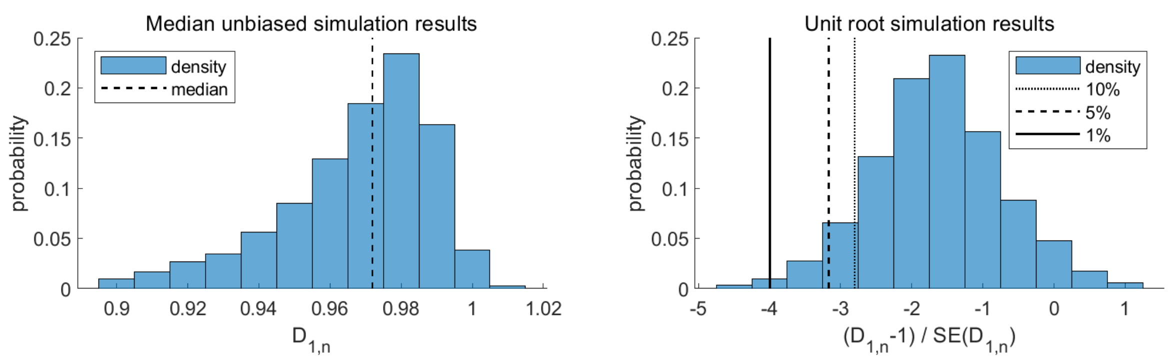

Figure 2: Simulation results used to obtain the median unbiased estimate of the largest eigenvalue of the VAR, and to test whether a unit root may be imposed on the VAR.

The fourth application obtains a median-unbiased estimate of the largest eigenvalue from the initial VAR, MU(D_1). Analogous to Andrews (1993) for an AR1, the aim is to find MU(D_1) that gives a simulated median value equal to D_1 from the initial VAR estimation. Hence, I set a trial value of MU(D_1), calculate the coefficients of the trial VAR using MU(D_1) and the remaining eigensystem parameters from the initial VAR estimation, and then simulate data from the trial VAR 10,000 times. For each simulation n, I subtract the mean of the simulated data, estimate the VAR on the mean-adjusted data, calculate its eigenvalues, and record $D_{1,n}$. Comparing the median for all simulations, i.e. $\text{Median}\left(\{D_{1,n}\}_{n=1}^{10{,}000}\right)$, to $D_1 = 0.9719$ from the initial VAR estimation, I then iterate with new trial values of MU(D_1) until the result is $\text{Median}\left(\{D_{1,n}\}_{n=1}^{10{,}000}\right) = 0.9719$. Panel 1 of figure 2 shows the results of the final trial that obtains the estimate MU(D_1) $= 0.9920$. Section 11 of the online appendix contains the VAR estimation results with the imposed constraint $D_1 =$ MU(D_1) $= 0.9920$ (which is not rejected statistically; $\mathcal{L}_1 = -668.50$, and $-2(\mathcal{L}_1 - \mathcal{L}_0) = 1.56$ with probability value $\chi^2(1.56, 1) = 0.788$). The half-life for D_1 in that model is 86.30 quarters, which illustrates the much greater persistence relative to the half-life of 24.30 quarters from the initial VAR.

The fifth application tests whether D_1 may be set to a unit root in the VAR. From the perspective of the AR1 processes for the EVAR components, the test is analogous to a Dickey-Fuller test. I use a simulation process as for the median-unbiased application, except the largest eigenvalue D_1 is set to 1 and only a single process of 10,000 simulations is required. For each simulation n, I obtain $D_{1,n}$ and its standard error $\text{SE}(D_{1,n})$ with OLS estimation of $X_{1,t}$ on $X_{1,t-1}$, and record the t-statistic $(D_{1,n} - 1) / \text{SE}(D_{1,n})$. Panel 2 of figure 2 shows the results, which has 10%, 5%, and 1% critical values of –2.80, –3.16, and –3.99 respectively. The t-statistic from the initial VAR estimation is $(D_1 - 1) / \text{SE}(D_1) = -1.83$, and so the null hypothesis of $D_1 = 1$ is not rejected at any standard level of significance. Section 12 of the online appendix contains the VAR estimation results with the constraint $D_1 = 1$. The vector $S_1 = [0.40, 0.78, 1]$ associated with the imposed eigenvalue $D_1 = 1$ is the estimated cointegrating vector.

5 Conclusion

This note shows how a VAR in its eigensystem form may be transformed into a system of AR1 processes, and the five illustrative applications demonstrate the inherent benefits of that perspective for VAR applications. Specifically, working with scalar processes is mathematically straightforward, and the statistical nature of an AR1 process is elementary and transparent.

Regarding extensions, the closed-form VAR forecast methods are readily applicable to impulse responses, by forecasting from an impulse vector $[\bar{y}_0, 0, \ldots, 0]$, and to closed-form FEV decompositions. Additionally, just one or several persistent components may be used for parsimonious long-horizon forecasts and FEVs, given that powers of smaller eigenvalues within the other components quickly converge to zero.

For VAR parsimony, the examples would be extended to systematic testing. For example, information criteria could be used to test increasing sets of zero eigenvalue constraints from the least to the most important EVAR components, e.g. $D_6 = 0$, then $D_5 = D_6 = 0$, and so on.

The median-unbiased estimation and unit root testing methods can be extended to the second (and subsequent) eigenvalues, e.g. calculating $\text{MU}(D_2)$ conditional on a VAR with $\text{MU}(D_1)$ already imposed, or testing $D_2 = 1$ conditional on $D_1 = 1$ already imposed (or jointly testing $D_1 = D_2 = 1$ with respect to the initial VAR). The estimation step in both could also be refined by constraining the eigensystem parameter/s that are not being estimated or tested to the estimated values from the initial VAR. The performance of median-unbiased VAR estimation remains to be assessed relative to other methods of bias correction, e.g. see Engsted and Pederson (2014), and VAR persistence imposition, e.g. Christiano (2012) summarizes Bayesian methods. The performance of VAR unit root testing, and cointegrating vector/VAR estimation within the EVAR component framework remains to be assessed relative to well-established approaches for those aspects; e.g. see Lütkepohl (2006) chapters 6-9 and Juselius (2018).

Beyond the focus on eigenvalues in this note, the unique parameters of the eigenvectors S_k associated with each AR1 process for D_k also offer a valuable perspective. A simple example is seeking further VAR parsimony by testing zero restrictions on immaterial S_k elements. More importantly, the inter-relationships among VAR variables from impulse responses may be considered and controlled using constraints on both S and D. For example, a constraint $S_{1,\pi} + c = S_{1,r}$ in the cointegrating vector S_1 associated with $D_1 = 1$ would impose a long-run real interest rate c, or setting $S_{k,u} = 0$ in the vector S_k would ensure that $\mathbb{E}_t[u_{t+h}]$ is unaffected by the dynamics associated with the AR1 process for D_k (e.g. $S_{1,u} = 0$ would be more consistent with economic principles). Appropriate combinations of constraints within S and D would therefore provide an avenue for identification within structural VARs. Repeated eigenvalue pair restrictions are particularly useful in this regard, because the exponential decay functions D_k^h in the associated impulse response component $S_k D_k^h X_{k,t} + S_{k+1} D_k^h X_{2,t} + S_k h D_k^{h-1} X_{k+1,t}$ allow an impulse to have instantaneous effects on selected variables, while the hump-shaped function hD_k^{h-1} allows selected variables to have only delayed responses.

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Online appendix for "Applications of vector autoregressions in their scalar autoregressive component form"

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4 November 2024

Abstract

This online appendix contains supporting and supplementary material related to comments and sections in the note "Applications of vector autoregressions in their scalar autoregressive component form".

JEL classification: C13, C32, C53

MOS classification: 62H12; 62H15; 62M10

Keywords: vector autoregression (VAR); companion matrix; eigenvalues; eigenvectors

1 Introduction

This appendix contains the following sections:

Section 2: Eigenvectors of a VAR companion matrix
Section 3: EVAR components for CCP eigenvalues
Section 4: VAR companion matrix in data form
Section 5: Wider context for the closed-form ergodic variance result
Section 6: Real AR2 forms for CCP eigenvalues
Section 7: EVAR components with repeated eigenvalues
Section 8: Complete set of ergodic variance results
Section 9: VAR estimates with a zero eigenvalue constraint
Section 10: VAR estimates with a repeated eigenvalue constraint
Section 11: VAR estimates with a median-unbiased eigenvalue constraint
Section 12: VAR estimates with a unit eigenvalue constraint

2 Eigenvectors of a VAR companion matrix

This section shows that the eigenvectors of a VAR companion matrix must take the forms given in section 2 of the main text.

Proposition 1 *Within the VAR companion matrix eigensystem decomposition, i.e.* $B = VDV^{-1}$, V *and equivalent expressions with an arbitrary normalization must take the following form:*

$$V_k = \begin{bmatrix} S_k D_k^{P-1} \\ \vdots \\ S_k D_k \\ S_k \end{bmatrix} \text{ with } S_k = \begin{bmatrix} S_{1,k} \\ \vdots \\ S_{N-1,k} \\ 1 \end{bmatrix} \text{ or } V = \begin{bmatrix} SD^{P-1} \\ \vdots \\ SD \\ S \end{bmatrix} \text{ and } V_X = \begin{bmatrix} S \\ SD^{-1} \\ \vdots \\ SD^{P-1} \end{bmatrix} \quad (1)$$

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where $S = [S_1, \ldots, S_{NP}]$. The block form for V in the third expression above is most general, accommodating Jordan blocks for repeated eigenvalues, while V_k within $V = [V_1, \ldots, V_{NP}]$ applies when eigenvalues are distinct. $V_X = VD^{1-P}$ is a convenient renormalization for the EVAR component framework.

Proof. The eigensystem decomposition $B = VDV^{-1}$ gives $BV = VD$. Writing the latter in full using the expression for B from section 2 of the main text and the block form for V in the third expression within equation 1 above gives:

$$\begin{bmatrix} \beta_{N \times NP} & \\ I_{NP-N} & 0_{[NP-N]\times N} \end{bmatrix} \begin{bmatrix} SD^{P-1} \\ SD^{P-2} \\ \vdots \\ SD \\ S \end{bmatrix} = \begin{bmatrix} SD^{P-1} \\ SD^{P-2} \\ \vdots \\ SD \\ S \end{bmatrix} D$$

$$\begin{bmatrix} \beta V \\ SD^{P-1} \\ \vdots \\ SD \\ S \end{bmatrix} = \begin{bmatrix} SD^{P} \\ SD^{P-1} \\ \vdots \\ SD \\ S \end{bmatrix} \tag{2}$$

All $N \times NP$ blocks are equalities apart from the first block $\beta V = SD^P$ that is an identity for the VAR coefficients; i.e. $\beta = SD^P V^{-1} = SD^{P-1} D V^{-1} = JVDV^{-1} = JB$. Therefore, V within $B = VDV^{-1}$ must have the block form given for V in equation 1.

The form for V_k in the case of distinct eigenvalues follows from $D = \text{diag}([D_1, \ldots, D_{NP}])$. Hence, each block of V will therefore be:

$$\begin{aligned} SD^p &= [S_1, \ldots, S_{NP}]\, \text{diag}\left([D_1, \ldots, D_{NP}]\right)^p \\ &= [S_1, \ldots, S_{NP}]\, \text{diag}\left(\left[D_1^p, \ldots, D_{NP}^p\right]\right) \\ &= \left[S_1 D_1^p, \ldots, S_{NP} D_{NP}^p\right] \end{aligned} \tag{3}$$

and so:

$$V = \begin{bmatrix} SD^{P-1} \\ SD^{P-2} \\ \vdots \\ SD \\ S \end{bmatrix} = \begin{bmatrix} S_1 D_1^{P-1} & \cdots & S_{NP} D_{NP}^{P-1} \\ S_1 D_1^{P-2} & \cdots & S_{NP} D_{NP}^{P-2} \\ \vdots & \cdots & \vdots \\ S_1 D_1 & \cdots & S_{NP} D_{NP} \\ S_1 & \cdots & S_{NP} \end{bmatrix} \tag{4}$$

Each column of V in the preceding expression is an eigenvector V_k in the form of the first expression within equation 1 above.

When D includes repeated eigenvalues, these are accommodated using Jordan blocks instead of only diagonal elements in the eigenvalue matrix D. Denoting individual Jordan blocks (including 1×1 blocks of distinct eigenvalues) with the subscript $[k]$, so $D = \text{diag}\left(\left[D_{[1]}, \ldots, D_{[K]}\right]\right)$, and partitioning S into a corresponding matrix $S = \left[S_{[1]}, \ldots, S_{[K]}\right]$, each $N \times NP$ row of V will be:

$$SD^p = \left[S_{[1]} D_{[1]}^p, \ldots, S_{[K]} D_{[K]}^p\right] \tag{5}$$

and so:

$$V = \begin{bmatrix} SD^{P-1} \\ SD^{P-2} \\ \vdots \\ SD \\ S \end{bmatrix} = \begin{bmatrix} S_{[1]} D_{[1]}^{P-1} & \cdots & S_{[K]} D_{[K]}^{P-1} \\ S_{[1]} D_{[1]}^{P-2} & \cdots & S_{[K]} D_{[K]}^{P-2} \\ \vdots & \cdots & \vdots \\ S_{[1]} D_{[1]} & \cdots & S_{[K]} D_{[K]} \\ S_{[1]} & \cdots & S_{[K]} \end{bmatrix} \tag{6}$$

Regarding the form of V_X, this follows from each block SD^{p-1} of V being multiplied by D^{1-P}, i.e.:

$$V_X = \begin{bmatrix} SD^{P-1} \\ SD^{P-2} \\ \vdots \\ SD \\ S \end{bmatrix} D^{1-P} = \begin{bmatrix} SD^{P-1}D^{1-P} \\ SD^{P-2}D^{1-P} \\ \vdots \\ SDD^{1-P} \\ SD^{1-P} \end{bmatrix} = \begin{bmatrix} S \\ SD^{-1} \\ \vdots \\ SD^{2-P} \\ SD^{1-P} \end{bmatrix} \tag{7}$$

■

3 EVAR components for CCP eigenvalues

This section shows, as stated in the introduction for section 3 of the main text, that CCP eigenvalues must be associated with CCP components. For completeness, I first establish the initial results that CCP eigenvalues must be associated with CCP columns of V and S, and CCP rows of V^{-1}. Those results are then applied in the context of CCP EVAR components.

Proposition 2 *CCP eigenvalues $(D_k, D_{k+1}) = \left(D_k, \overline{D_k}\right)$ are associated with CCP eigenvectors $(V_k, V_{k+1}) = \left(V_k, \overline{V_k}\right)$, CCP vectors $(S_k, S_{k+1}) = \left(S_k, \overline{S_k}\right)$, and CCP rows k and $k+1$ in the inverse of the eigenvector matrix V^{-1}.*

Proof. For notational convenience, first consider a specific case where there is only a single pair of complex conjugate eigenvalues and they are arranged to be the first two entries, i.e. $(D_1, D_2) = \left(D_1, \overline{D_1}\right)$ and $D = \text{diag}\left(\left[D_1, \overline{D_1}, D_3, \ldots, D_{NP}\right]\right)$. For each eigenvector $BV_k = V_k D_k$, so $BV_1 = V_1 D_1$ and $BV_2 = V_2 D_2$. Then $\overline{BV_2} = \overline{V_2 D_2}$, so $B\overline{V_2} = \overline{V_2} \times \overline{D_2} = \overline{V_2} D_1$, therefore $\overline{V_2} = V_1$, and $V_2 = \overline{V_1}$.

Regarding (S_1, S_2), these are the last N rows of (V_1, V_2). Therefore, given $V_2 = \overline{V_1}$, then $S_2 = \overline{S_1}$.

To establish that the first two rows of V^{-1} are a CCP, first define a block-diagonal permutation matrix A as $A = \text{diag}([A_2, I_{NP-2}])$, where I_{NP-2} is the $(NP-2) \times (NP-2)$ identity matrix, and A_2 is:

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A_2^{-1} \tag{8}$$

and note that $A_2 = A_2^{-1}$ is apparent from A_2^2 being the 2×2 identity matrix. The product VA interchanges the first two columns of V, so $VA = \left[\overline{V_1}, V_1, V_3, \ldots, V_{NP}\right] = \overline{V}$. Taking the inverse gives:

$$\begin{aligned} (VA)^{-1} &= \left(\overline{V}\right)^{-1} \\ A^{-1}V^{-1} &= \overline{V^{-1}} \\ AV^{-1} &= \overline{V^{-1}} \end{aligned} \tag{9}$$

The product AV^{-1} interchanges the first two rows of V^{-1}, so $AV^{-1} = \overline{V^{-1}}$ establishes that the first two rows of V^{-1} are a CCP. That is, in full with $\left[V^{-1}\right]_k$ denoting each row of V^{-1}:

$$\begin{aligned} AV^{-1} &= \overline{V^{-1}} \\ A \begin{bmatrix} \left[V^{-1}\right]_1 \\ \left[V^{-1}\right]_2 \\ \left[V^{-1}\right]_3 \\ \vdots \\ \left[V^{-1}\right]_{NP} \end{bmatrix} &= \overline{\begin{bmatrix} \left[V^{-1}\right]_1 \\ \left[V^{-1}\right]_2 \\ \left[V^{-1}\right]_3 \\ \vdots \\ \left[V^{-1}\right]_{NP} \end{bmatrix}} \end{aligned} \tag{10}$$

hence:

$$\begin{bmatrix} \left[V^{-1}\right]_2 \\ \left[V^{-1}\right]_1 \\ \left[V^{-1}\right]_3 \\ \vdots \\ \left[V^{-1}\right]_{NP} \end{bmatrix} = \begin{bmatrix} \overline{\left[V^{-1}\right]_1} \\ \overline{\left[V^{-1}\right]_2} \\ \overline{\left[V^{-1}\right]_3} \\ \vdots \\ \overline{\left[V^{-1}\right]_{NP}} \end{bmatrix}$$

so $\left[V^{-1}\right]_2 = \overline{\left[V^{-1}\right]_1}$ and $\left[V^{-1}\right]_1 = \overline{\left[V^{-1}\right]_2}$. For the remaining rows, $\left[V^{-1}\right]_k = \overline{\left[V^{-1}\right]_k}$ means those rows must be real.

When D contains more than a single set of CCP eigenvalues, the procedure above is applied to each eigenvalue pair $(D_k, D_{k+1}) = \left(D_k, \overline{D_k}\right)$. Hence $\left(D_k, \overline{D_k}\right)$ will be associated with $\left(V_k, \overline{V_k}\right)$ and $\left(S_k, \overline{S_k}\right)$ and, using a permutation matrix A with A_2 at each $(k, k+1)$ block-diagonal entry, $(V_k, V_{k+1}) = \left(V_k, \overline{V_k}\right)$ will be associated with $\left[V^{-1}\right]_{k+1} = \overline{\left[V^{-1}\right]_k}$. ■

Proposition 3 *For CCP eigenvalues, the components* $(X_{k,t}, X_{k+1,t})$ *and* $(S_k X_{k,t}, S_{k+1} X_{k+1,t})$*, and their forecasts* $\left(D_k^h X_{k,t}, D_k^h X_{k+1,t}\right)$ *and* $\left(S_k D_k^h X_{k,t}, S_{k+1} D_k^h X_{k+1,t}\right)$ *will all be CCPs.*

Proof. $X_t = V_X^{-1} = D^{P-1} V^{-1} \bar{Y}_t$, and so $(X_{k,t}, X_{k+1,t})$ associated with a 2×2 CCP eigenvalue block will be:

$$\begin{aligned} \begin{bmatrix} X_{k,t} \\ X_{k+1,t} \end{bmatrix} &= \begin{bmatrix} D_k & 0 \\ 0 & \overline{D_k} \end{bmatrix}^{P-1} \begin{bmatrix} \left[V^{-1}\right]_1 \\ \overline{\left[V^{-1}\right]_1} \end{bmatrix} \bar{Y}_t \\ &= \begin{bmatrix} D_k^{P-1} \left[V^{-1}\right]_1 \bar{Y}_t \\ \overline{D_k^{P-1} \left[V^{-1}\right]_k \bar{Y}_t} \end{bmatrix} \\ &= \begin{bmatrix} D_1^{P-1} \left[V^{-1}\right]_1 \bar{Y}_t \\ \overline{D_k^{P-1} \left[V^{-1}\right]_1 \bar{Y}_t} \end{bmatrix} \end{aligned} \tag{11}$$

so $(X_{k,t}, X_{k+1,t}) = \left(X_{k,t}, \overline{X_{k,t}}\right)$. Given this result, then:

$$\begin{aligned} (S_k X_{k,t}, S_{k+1} X_{k+1,t}) &= \left(S_k X_{k,t}, \overline{S_k} \times \overline{X_{k,t}}\right) = \left(S_k X_{k,t}, \overline{S_k X_{k,t}}\right) \\ \left(D_k^h X_{k,t}, D_{k+1}^h X_{k+1,t}\right) &= \left(D_k^h X_{k,t}, \overline{D_k^h} \times \overline{X_{k,t}}\right) = \left(D_k^h X_{k,t}, \overline{D_k^h X_{k,t}}\right) \\ \left(S_k D_k^h X_{k,t}, S_{k+1} D_{k+1}^h X_{k+1,t}\right) &= \left(S_k D_k^h X_{k,t}, \overline{S_k} \overline{D_k^h} \overline{X_{k,t}}\right) = \left(S_k D_k^h X_{k,t}, \overline{S_k D_k^h X_{k,t}}\right) \end{aligned} \tag{12}$$

■

4 VAR companion matrix in data form

This section contains material related to Proposition 2 of the main text, i.e. first a proof that the estimate of the VAR coefficient matrix from the data is $\beta = y \bar{Y}_L' \left(\bar{Y}_L' \bar{Y}_L\right)^{-1}$, and then a proof that the VAR companion matrix may be expressed as $B = \bar{Y} \bar{Y}_L' \left(\bar{Y}_L \bar{Y}_L'\right)^{-1}$.

Proposition 4 *Following Hamilton (1994) pp. 293-96, the summation form the log-likelihood function for a VAR conditioned on the initial* P *observations of a given* $N \times (P+T)$ *dataset* $\{\bar{y}_t\}_{1-P}^{T}$*, is:*

$$\mathcal{L}\left(\left[\beta, \Omega_\varepsilon\right], \{\bar{y}_t\}_{1-P}^{T}\right) = -\frac{NT}{2} \log(2\pi) - \frac{T}{2} \log\left(\det\left[\Omega_\varepsilon\right]\right) - \frac{1}{2} \sum_{t=1}^{T} \varepsilon_t' \Omega_\varepsilon^{-1} \varepsilon_t \tag{13}$$

where $\varepsilon_t = \bar{y}_t - \beta \bar{Y}_{t-1}$*. Maximizing this log-likelihood function results in the VAR coefficient matrix estimate* $\beta = \bar{y}\bar{Y}_L' \left(\bar{Y}_L'\bar{Y}_L\right)^{-1}$ *or* $\beta_{[n]} = \bar{y}_{[n]} Y_L' \left(Y_L Y_L'\right)^{-1}$ *for row* n *of* β*, where* $\bar{y}$ *is an* $N \times T$ *matrix of the* $\bar{y}_t$ *data, and* $\bar{Y}_L$ *is an* $NP \times T$ *matrix of the* $\bar{Y}_{t-1}$ *data. The VAR variance matrix estimate is* $\Omega_\varepsilon = \frac{1}{T}\varepsilon\varepsilon'$*, where* ε *is the* $N \times T$ *matrix of the residuals* ε_t*.*

Proof. Equation 13 may be equivalently expressed as:

$$\mathcal{L}(\cdot) = -\frac{NT}{2}\log(2\pi) - \frac{T}{2}\log\left(\det\left[\Omega_\varepsilon\right]\right) - \frac{1}{2}\mathrm{tr}\left[\left(\bar{y} - \beta\bar{Y}_L\right)'\Omega_\varepsilon^{-1}\left(\bar{y} - \beta\bar{Y}_L\right)\right] \tag{14}$$

where tr[·] is the trace operator. To see the equivalence of both expressions, note that each $\varepsilon_t'\Omega_\varepsilon^{-1}\varepsilon_t = \left(\bar{y}_t - \beta\bar{Y}_{t-1}\right)'\Omega_\varepsilon^{-1}\left(\bar{y}_t - \beta\bar{Y}_{t-1}\right)$ in equation 13 gives a scalar result for the given time t, and the summation adds the results for all times from 1 to T. Within equation 14, $\left(\bar{y} - \beta\bar{Y}_L\right)'\Omega_\varepsilon^{-1}\left(\bar{y} - \beta\bar{Y}_L\right)$ creates a $T \times T$ matrix with the results $\left(\bar{y}_t - \beta\bar{Y}_{t-1}\right)'\Omega_\varepsilon^{-1}\left(\bar{y}_t - \beta\bar{Y}_{t-1}\right)$ on the diagonal, and those are summed using the trace operator.

Expanding the expression within tr[·] gives:

$$\begin{aligned}
& \mathrm{tr}\left[\left(\bar{y} - \beta\bar{Y}_L\right)'\Omega_\varepsilon^{-1}\left(\bar{y} - \beta\bar{Y}_L\right)\right] \\
= \; & \mathrm{tr}\left[\left(\bar{y}' - \bar{Y}_L'\beta'\right)\Omega_\varepsilon^{-1}\left(\bar{y} - \beta\bar{Y}_L\right)\right] \\
= \; & \mathrm{tr}\left[\bar{y}'\Omega_\varepsilon^{-1}\bar{y} - \bar{y}'\Omega_\varepsilon^{-1}\beta\bar{Y}_L - \bar{Y}_L'\beta'\Omega_\varepsilon^{-1}\bar{y} + \bar{Y}_L'\beta'\Omega_\varepsilon^{-1}\beta\bar{Y}_L\right] \\
= \; & \mathrm{tr}\left[\bar{y}'\Omega_\varepsilon^{-1}\bar{y} - 2\bar{Y}_L'\beta'\Omega_\varepsilon^{-1}\bar{y} + \bar{Y}_L'\beta'\Omega_\varepsilon^{-1}\beta\bar{Y}_L\right]
\end{aligned} \tag{15}$$

where the last line combines $\bar{y}'\Omega_\varepsilon^{-1}\beta\bar{Y}_L$ and $\bar{Y}_L'\beta'\Omega_\varepsilon^{-1}\bar{y}$, given that $\mathrm{tr}[A'] = \mathrm{tr}[A]$ where A is a generic square matrix.

To find the matrix β that maximizes $\mathcal{L}(\cdot)$, differentiate $\mathcal{L}(\cdot)$ with respect to β' and set the result to zero, i.e.:

$$\begin{aligned}
0 &= \frac{\partial}{\partial\beta'}\mathcal{L}(\cdot) \\
&= -0 - 0 + \frac{\partial \mathrm{tr}\left[\bar{y}'\Omega_\varepsilon^{-1}\bar{y} - 2\bar{Y}_L'\beta'\Omega_\varepsilon^{-1}\bar{y} + \bar{Y}_L'\beta'\Omega_\varepsilon^{-1}\beta\bar{Y}_L\right]}{\partial\beta'} \\
&= -2\Omega_\varepsilon^{-1}\bar{y}Y_L' + 2\Omega_\varepsilon^{-1}\beta\bar{Y}_L\bar{Y}_L' \\
\beta\bar{Y}_L\bar{Y}_L' &= \bar{y}Y_L' \\
\beta &= y\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}
\end{aligned} \tag{16}$$

where the third line uses two matrix calculus results,[1] i.e. for generic matrices A, B, C, and X:

$$\begin{aligned}
\frac{\partial \mathrm{tr}(AXB)}{\partial X} &= BA \\
\frac{\partial \mathrm{tr}(AXBX'C)}{\partial X} &= BX'CA + B'X'A'C'
\end{aligned} \tag{17}$$

The form for equation 16 makes it clear that each row β could be estimated by a separate OLS regression of $\bar{y}_t$ on $\bar{Y}_{t-1}$. That is, each row n of $\beta = y\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$ is $\beta_{[n]} = \bar{y}_{[n]}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$, where $\bar{y}_{[n]} = \left[\bar{y}_{n,1}, \ldots, \bar{y}_{n,T}\right]$ is the $1 \times T$ vector of data for variable n. That expression is an OLS regression of $\bar{y}_{[n]}$ on all of the lagged data.

[1] See Petersen and Pedersen (2012), which also contains the matrix calculus results subsequently used for Ω_ε. I use the numerator layout convention.

To find the variance matrix Ω_ε:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \Omega_\varepsilon} \mathcal{L}(\cdot) \\
&= -\frac{T}{2}\frac{\partial}{\partial \Omega_\varepsilon} \log\left(\det\left[\Omega_\varepsilon\right]\right) - \frac{1}{2}\mathrm{tr}\left[\varepsilon' \Omega_\varepsilon^{-1} \varepsilon\right] \\
&= -T\Omega_\varepsilon^{-1} + \varepsilon\varepsilon'\Omega_\varepsilon^{-2} \\
T\Omega_\varepsilon &= \varepsilon\varepsilon' \\
\Omega_\varepsilon &= \frac{1}{T}\varepsilon\varepsilon'
\end{aligned} \tag{18}$$

where the second and third lines respectively make use of the following results for generic matrices A and B and an invertible square matrix X:

$$\frac{\partial \log\left(\det\left[X\right]\right)}{\partial X} = X^{-1} \tag{19}$$

$$\begin{aligned}
\frac{\partial \mathrm{tr}\left(AX^{-1}B\right)}{\partial X} &= \frac{\partial \mathrm{tr}\left(AX^{-1}B\right)}{\partial X^{-1}} \frac{\partial X^{-1}}{\partial X} \\
&= BA \times -X^{-2} \\
&= -BAX^{-2}
\end{aligned} \tag{20}$$

■

Proposition 5 *The VAR companion matrix B may be expressed as $B = \bar{Y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$.*

Proof. The matrices of data and lagged data, $\bar{Y}$ and $\bar{Y}_L$, may be expressed in partitioned form as:

$$\bar{Y} = \begin{bmatrix} \bar{y} \\ \bar{y}_{-1} \\ \vdots \\ \bar{y}_{-P+1} \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \bar{Y}_{L*} \end{bmatrix} \; ; \bar{Y}_L = \begin{bmatrix} \bar{y}_{-1} \\ \vdots \\ \bar{y}_{-P+1} \\ \bar{y}_{-P} \end{bmatrix} = \begin{bmatrix} \bar{Y}_{L*} \\ \bar{y}_{-P} \end{bmatrix} \tag{21}$$

where $\bar{y}$ is the $N \times T$ matrix of data y_t for each period, each $\bar{y}_{-p}$ is the $N \times T$ matrix of lagged data $\bar{Y}_{t-p}$, and $\bar{Y}_{L*} = \left[\bar{y}_{-1}', \ldots, \bar{y}_{-P+1}'\right]'$, an $[NP - N] \times T$ matrix.

$\bar{Y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$ may therefore be expressed in partitioned form as:

$$\begin{aligned}
\bar{Y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} &= \begin{bmatrix} \bar{y} \\ \bar{Y}_{L*} \end{bmatrix} \bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} \\
&= \begin{bmatrix} \bar{y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} \\ \bar{Y}_{L*}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \beta \\ \bar{Y}_{L*}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} \end{bmatrix}
\end{aligned} \tag{22}$$

where $\beta = \bar{y}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$. The bottom block $\bar{Y}_{L*}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$ is:

$$\begin{aligned}
\bar{Y}_{L*}\bar{Y}_L' \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} &= \bar{Y}_{L*}\left[\bar{Y}_{L*}', \bar{y}_{-P}'\right] \left(\bar{Y}_L\bar{Y}_L'\right)^{-1} \\
&= \left[\bar{Y}_{L*}\bar{Y}_{L*}', \bar{Y}_{L*}\bar{y}_{-P}'\right] \left(\bar{Y}_L\bar{Y}_L'\right)^{-1}
\end{aligned} \tag{23}$$

$\bar{Y}_L\bar{Y}_L'$ in partitioned form is:

$$\begin{aligned} \bar{Y}_L\bar{Y}_L' &= \begin{bmatrix} \bar{Y}_{L*} \\ \bar{y}_{-P} \end{bmatrix} \left[\bar{Y}_{L*}', \bar{y}_{-P}'\right] \\ &= \begin{bmatrix} \bar{Y}_{L*}\bar{Y}_{L*}' & \bar{Y}_{L*}\bar{y}_{-P}' \\ \left(\bar{Y}_{L*}\bar{y}_{-P}'\right)' & \bar{y}_{-P}\bar{y}_{-P}' \end{bmatrix} \end{aligned} \tag{24}$$

and $\bar{Y}\bar{Y}_L$ in partitioned form is:

$$\begin{aligned} \bar{Y}\bar{Y}_L &= \begin{bmatrix} \bar{y} \\ \bar{Y}_{L*} \end{bmatrix} \left[\bar{Y}_{L*}', \bar{y}_{-P}'\right] \\ &= \begin{bmatrix} \bar{y}\left[\bar{Y}_{L*}', \bar{y}_{-P}'\right] \\ \bar{Y}_{L*}\left[\bar{Y}_{L*}', \bar{y}_{-P}'\right] \end{bmatrix} \\ &= \begin{bmatrix} \bar{y}\bar{Y}_L' \\ \left[\bar{Y}_{L*}\bar{Y}_{L*}', \bar{Y}_{L*}\bar{y}_{-P}'\right] \end{bmatrix} \end{aligned} \tag{25}$$

$\bar{Y}_{L*}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$ in partitioned form is therefore:

$$\begin{aligned} \bar{Y}_{L*}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1} &= \left[\bar{Y}_{L*}\bar{Y}_{L*}', \bar{Y}_{L*}\bar{y}_{-P}'\right] \begin{bmatrix} \bar{Y}_{L*}\bar{Y}_{L*}' & \bar{Y}_{L*}\bar{y}_{-P}' \\ \left(\bar{Y}_{L*}\bar{y}_{-P}'\right)' & \bar{y}_{-P}\bar{y}_{-P}' \end{bmatrix}^{-1} \\ &= [A, B] \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \end{aligned} \tag{26}$$

where A, B, C, and D in the present context are simply generic labels for the matrices to make the partitioned form for $\bar{Y}_{L*}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$ more apparent. In particular, equation 26 may be seen as the two top blocks of the identity:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I_{\dim(A)} & 0_{\dim(A)\times\dim(D)} \\ 0_{\dim(D)\times\dim(A)} & I_{\dim(D)} \end{bmatrix} \tag{27}$$

where $I_{\dim(A)}$ is the identity matrix, $0_{\dim(A)\times\dim(D)}$ is a matrix of zeros, and $\dim(\cdot)$ gives the relevant dimensions for each matrix in terms of the dimensions of the square matrices A and D. Alternatively, the explicit expression for the inverse of a partitioned matrix may be used, i.e.:

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \begin{bmatrix} A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1}CA^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^{-1} \\ -\left(D - CA^{-1}B\right)^{-1}CA^{-1} & \left(D - CA^{-1}B\right)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} + A^{-1}BFCA^{-1} & -A^{-1}BF \\ -FCA^{-1} & F \end{bmatrix} \end{aligned} \tag{28}$$

where $F = \left(D - CA^{-1}B\right)^{-1}$. Therefore:

$$\begin{aligned} [A, B] \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \left[I_{\dim(A)} + BFCA^{-1} - BFCA^{-1}, -BF + BF\right] \\ &= \left[I_{\dim(A)}, 0_{\dim(A)\times\dim(D)}\right] \end{aligned} \tag{29}$$

Hence, with the substitutions $A = \bar{Y}_{L*}\bar{Y}_{L*}'$ and $D = \bar{y}_{-P}\bar{y}_{-P}'$, the expression $\bar{Y}_{L*}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}$ is:

$$\begin{aligned} \bar{Y}_{L*}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1} &= \left[I_{\dim\left(\bar{Y}_{L*}\bar{Y}_{L*}'\right)}, 0_{\dim\left(\bar{Y}_{L*}\bar{Y}_{L*}'\right)\times\dim\left(\bar{y}_{-P}\bar{y}_{-P}'\right)}\right] \\ &= \left[I_{NP-N}, 0_{[NP-N]\times N}\right] \end{aligned} \tag{30}$$

and therefore:

$$\bar{Y}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}=\left[\begin{array}{c}\beta\\ \bar{Y}_{L*}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}\end{array}\right]=\left[\begin{array}{cc}\multicolumn{2}{c}{\bar{y}\bar{Y}_L'\left(\bar{Y}_L\bar{Y}_L'\right)^{-1}}\\ I_{NP-N} & 0_{[NP-N]\times N}\end{array}\right]=B \tag{31}$$

■

5 Wider context for the closed-form ergodic variance result

This section provides additional context and discussion for the ergodic variance result contained in section 3.2 of the main text.

The ergodic variance result $\Omega_y(\infty)$ from section 3.2 is based on a solution to the discrete-time Lyapunov equation. Using the notation from the main text, the Lyapunov equation is $B\Omega_Y(\infty)B+\Omega_{E_Y}=\Omega_Y(\infty)$ and its solution is the $NP\times NP$ matrix $\Omega_Y(\infty)=V_X\Omega_X(\infty)V_X^{\dagger}$, from which the $N\times N$ matrix $\Omega_y(\infty)=J\Omega_Y(\infty)J'$ is obtained.

The more general context for the Lyapunov equation is solving for Ω_A in $A\Omega_AA+\Omega=\Omega_A$ where the generic square matrix A and the generic symmetric matrix Ω may be dense, unlike the sparse matrices B and Ω_{E_Y} for a VAR with more than a single lag. As discussed in Doan (2010) sections 4 and 5, there are a variety of methods for solving the general Lyapunov equation. The method based on vectorization is:

$$\text{vec}\left(\Omega_A\right)=\left(I_{M^2}-A\otimes A\right)^{-1}\text{vec}\left(\Omega\right) \tag{32}$$

where M is the dimension of A, Ω_A, and Ω. This expression is often presented in econometrics textbooks, e.g. see Hamilton (1994) p. 265 and Lütkepohl (2006) eq. 2.1.39, but it involves matrices of dimension M^2. Doan (2010) highlights the computational inefficiency of the vectorization method, given the solution requires $O\left(M^6\right)$ arithmetic operations, whereas methods that retain the original matrix dimensions, e.g. Kitagawa (1977) and Johansen (2002), require $O\left(M^3\right)$ operations (including the allowance for eigensystem or Schur decompositions).

The method developed in the main text is therefore within the efficient class of Lyapunov equation solutions. It also has the relative advantage of intuition, i.e. the method used to obtain the solution is the infinite limit of a finite solution. The example in the context of VARs from the main text is obtaining the VAR ergodic variance $\Omega_Y(\infty)$ as the limit of the finite-horizon FEV expression $\Omega_Y(h)$. In this respect, my solution method parallels the continuous-time solutions for $\Omega_Y(\infty)$ and $\Omega_Y(h)$ developed in Rome (1969) for the first-order multivariate stochastic differential equation (which accommodates higher-order equations by using a companion matrix). The parallels are inherent, given VARs are a discrete-time analogue of the continuous-time case; i.e. a VAR is a equivalent to to first-order multivariate stochastic difference equation (which also accommodates higher-order differences by using a companion matrix).

One further observation related to the ergodic variance $\Omega_Y(\infty)$ and the finite solution $\Omega_Y(h)$ is that the FEV for a VAR may be equivalently expressed as $\Omega_Y(\infty)$ and $\Omega_Y(h)$ relative to $\Omega_X(\infty)$, i.e.:

$$\Omega_Y(h)=\Omega_Y(\infty)+\left[\Omega_Y(h)-\Omega_Y(\infty)\right] \tag{33}$$

where $\left[\Omega_Y(h)-\Omega_Y(\infty)\right]$ may be calculated as $V_X\left[\Omega_X(h)-\Omega_X(\infty)\right]V_X^{\dagger}$, and the elements of $\left[\Omega_X(h)-\Omega_X(\infty)\right]$ are:

$$\left[\Omega_X(h)\right]_{ij}-\left[\Omega_X(\infty)\right]_{ij}=-\Omega_{E_X,ij}\frac{\left(D_i\overline{D_j}\right)^h}{1-D_i\overline{D_j}} \tag{34}$$

Therefore, regardless of the method used to obtain $\Omega_Y(\infty)$, that result may be used in conjunction with the closed-form adjustment $\left[\Omega_Y(h)-\Omega_Y(\infty)\right]$ to obtain $\Omega_Y(h)$ for an arbitrary horizon.

6 Real AR2 forms for CCP eigenvalues

This section provides details related to my comments in section 3.3 of the main text that the AR1 processes and components associated with CCP eigenvalues could be expressed in real forms based on the AR2 companion matrix, but why it is preferrable to use AR1 processes and forms where possible.

I first show how eigenvalue pairs may be expressed using the AR2 companion matrix, and then use that result to re-express CCP AR1 processes and their associated forecasts in real AR2 form. However, the point forecast expressions for the latter require powers of the AR2 companion matrix, which does not have a closed-form solution based on a scalar powers (unless converted back to AR1 form). An alternative is to express the point forecasts in terms of real scalar cosine and sine functions, as shown in Proposition 8. While useful for exposition, this would be unwieldy for applications, and more so if extended to variance expressions. Therefore, Proposition 9 provides the FEVs and ergodic variances for CCPs of EVAR components on the basis of AR1 components.

Proposition 6 *For distinct eigenvalues, the AR2 companion matrix $\Phi_{[k]}$ is related to the 2×2 diagonal eigenvalue matrix as follows:*

$$D_{[k]} = U_{[k]}^{-1} \Phi_{[k]} U_{[k]} \tag{35}$$

where:

$$D_{[k]} = \begin{bmatrix} D_k & 0 \\ 0 & D_{k+1} \end{bmatrix} ; U_{[k]} = \begin{bmatrix} D_k & D_{k+1} \\ 1 & 1 \end{bmatrix} ; \Phi_{[k]} = \begin{bmatrix} \phi_k & \phi_{k+1} \\ 1 & 0 \end{bmatrix} \tag{36}$$

with (D_k, D_{k+1}) a real eigenvalue pair or CCP eigenvalues. Powers of $D_{[k]}$ may be expressed as:

$$D_{[k]}^h = U_{[k]}^{-1} \Phi_{[k]}^h U_{[k]} \tag{37}$$

Proof. The companion matrix of a scalar AR2 with distinct eigenvalues may be expressed as the following eigensystem decomposition:

$$\begin{bmatrix} \phi_k & \phi_{k+1} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} D_k & D_{k+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} D_k & 0 \\ 0 & D_{k+1} \end{bmatrix} \begin{bmatrix} D_k & D_{k+1} \\ 1 & 1 \end{bmatrix}^{-1} \tag{38}$$

which is evident by direct evaluation, i.e.:

$$\begin{aligned} \begin{bmatrix} \phi_k & \phi_{k+1} \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} D_k & D_{k+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} D_k & 0 \\ 0 & D_{k+1} \end{bmatrix} \frac{1}{D_k - D_{k+1}} \begin{bmatrix} 1 & -D_{k+1} \\ -1 & D_k \end{bmatrix} \\ &= \frac{1}{D_k - D_{k+1}} \begin{bmatrix} D_k^2 & D_{k+1}^2 \\ D_k & D_{k+1} \end{bmatrix} \begin{bmatrix} 1 & -D_{k+1} \\ -1 & D_k \end{bmatrix} \\ &= \frac{1}{D_k - D_{k+1}} \begin{bmatrix} D_k^2 - D_{k+1}^2 & D_k D_{k+1}^2 - D_k^2 D_{k+1} \\ D_k - D_{k+1} & 0 \end{bmatrix} \\ &= \frac{1}{D_k - D_{k+1}} \begin{bmatrix} (D_k - D_{k+1})(D_k + D_{k+1}) & -(D_k - D_{k+1}) D_k D_{k+1} \\ D_k - D_{k+1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} D_k + D_{k+1} & -D_k D_{k+1} \\ 1 & 0 \end{bmatrix} \end{aligned} \tag{39}$$

so $\phi_k = D_k + D_{k+1}$ and $\phi_{k+1} = -D_k D_{k+1}$ which, from Hamilton (1994) p. 30, are AR2 coefficients expressed in terms of distinct AR2 eigenvalues.

Powers of $D_{[k]}^h$ are then:

$$\begin{aligned}
D_{[k]}^h &= \left(U_{[k]}^{-1}\Phi_{[k]}U_{[k]}\right)^h \\
&= U_{[k]}^{-1}\Phi_{[k]}U_{[k]} \times U_{[k]}^{-1}\Phi_{[k]}U_{[k]} \times \ldots \times U_{[k]}^{-1}\Phi_{[k]}U_{[k]} \\
&= U_{[k]}^{-1}\Phi_{[k]}^h U_{[k]} \qquad (40)
\end{aligned}$$

■

Proposition 7 *For CCP eigenvalues $(D_k, D_{k+1}) = \left(D_k, \overline{D_k}\right)$, the associated EVAR processes, their forecasts, and their component forecasts may be expressed in real form based on the AR2 companion matrix.*

Proof. For CCP eigenvalues, the process for $[X_{k,t}, X_{k+1,t}]'$ is:

$$X_{[k],t} = D_{[k]}X_{[k],t-1} + E_{X,[k],t} \qquad (41)$$

where:

$$X_{[k],t} = \begin{bmatrix} X_{k,t} \\ \overline{X_{k,t}} \end{bmatrix} ;\ D_{[k]} = \begin{bmatrix} D_k & 0 \\ 0 & \overline{D_k} \end{bmatrix} ;\ X_{[k],t-1} = \begin{bmatrix} X_{k,t-1} \\ \overline{X_{k,t-1}} \end{bmatrix} ;\ E_{X,[k],t} = \begin{bmatrix} E_{X,k,t} \\ \overline{E_{X,k,t}} \end{bmatrix} \qquad (42)$$

which may be transformed to AR2 form as follows:

$$\begin{aligned}
X_{[k],t} &= D_{[k]}X_{[k],t-1} + E_{X,[k],t} \\
&= U_{[k]}^{-1}\Phi_{[k]}U_{[k]}X_{[k],t-1} + E_{X,[k],t} \\
U_{[k]}X_{[k],t} &= \Phi_{[k]}U_{[k]}X_{[k],t-1} + U_{[k]}E_{X,[k],t} \\
Z_{[k],t} &= \Phi_{[k]}Z_{[k],t-1} + E_{Z,[k],t} \qquad (43)
\end{aligned}$$

$Z_{[k],t}$ will be real, given:

$$\begin{aligned}
U_{[k]}X_{[k],t} &= \begin{bmatrix} D_k & \overline{D_k} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_{k,t} \\ \overline{X_{k,t}} \end{bmatrix} \\
&= \left[2\,\mathrm{Re}\left(D_k X_{k,t}\right), 2\,\mathrm{Re}\left(X_{k,t}\right)\right] \qquad (44)
\end{aligned}$$

and likewise $Z_{[k],t-1} = U_{[k]}X_{[k],t-1}$ and $E_{Z,[k],t} = U_{[k]}E_{X,[k],t}$ will also be real.

Using $D_{[k]}^h = U_{[k]}^{-1}\Phi_{[k]}^h U_{[k]}$, forecasts of $X_{[k],t}$ may be expressed in real AR2 form, i.e.:

$$\begin{aligned}
\mathbb{E}_t\left[X_{[k],t+h}\right] &= D_{[k]}^h X_{[k],t} \\
&= U_{[k]}^{-1}\Phi_{[k]}^h U_{[k]}X_{[k],t} \\
\mathbb{E}_t\left[U_{[k]}X_{[k],t}\right] &= \Phi_{[k]}U_{[k]}X_{[k],t} \\
Z_{[k],t+h} &= \Phi_{[k]}^h Z_{[k],t} \qquad (45)
\end{aligned}$$

where $Z_{[k],t+h} = U_{[k]}X_{[k],t+h}$ will be real, as for $Z_{[k],t} = U_{[k]}X_{[k],t}$ earlier.

The sum of CCP forecast components may be expressed in real AR2 form, i.e.:

$$\begin{aligned}
& S_k D_k^h X_{k,t} + \overline{S_k D_k^h X_{k,t}} \\
&= \left[S_k, \overline{S_k}\right] \begin{bmatrix} D_k^h & 0 \\ 0 & \overline{D_k}^h \end{bmatrix} \begin{bmatrix} X_{k,t} \\ \overline{X_{k,t}} \end{bmatrix} \\
&= \left[S_k, \overline{S_k}\right] D_{[k]}^h X_{[k],t} \\
&= \left[S_k, \overline{S_k}\right] U_{[k]}^{-1}\Phi_{[k]}^h U_{[k]}X_{[k],t} \\
&= \left[R_k, R_{k+1}\right] \Phi_{[k]}^h Z_{[k],t} \qquad (46)
\end{aligned}$$

where $[R_k, R_{k+1}]$ will be real given:

$$U_{[k]}^{-1} = \frac{1}{2\operatorname{Im}(D_k)} \begin{bmatrix} 1 & -\overline{D_k} \\ -1 & D_k \end{bmatrix} \tag{47}$$

and so:

$$\begin{aligned} \left[S_k, \overline{S_k}\right] U_{[k]}^{-1} &= \frac{1}{2\operatorname{Im}(D_k)} \left[S_k, \overline{S_k}\right] \begin{bmatrix} 1 & -\overline{D_k} \\ -1 & D_k \end{bmatrix} \\ &= \frac{1}{2\operatorname{Im}(D_k)} \left[2\operatorname{Im}(S_k), 2\operatorname{Im}(D_k S_k)\right] \end{aligned} \tag{48}$$

■

Proposition 8 *CCP forecast components may be expressed as real trigonometric functions, i.e.:*

$$D_k^h X_{k,t} + \overline{D_k^h X_{k,t}} = 2r_k^h \cos(h\theta_k) \operatorname{Re}(X_{k,t}) - 2r_k^h \sin(h\theta_k) \operatorname{Im}(X_{k,t}) \tag{49}$$

and:

$$\begin{aligned} & S_k D_k^h X_{k,t} + \overline{S_k D_k^h X_{k,t}} \\ = & \operatorname{Re}(S_k) \cdot 2r_k^h \left[\cos(h\theta_k) \operatorname{Re}(X_{k,t}) - \sin(h\theta_k) \operatorname{Im}(X_{k,t})\right] \\ & - \operatorname{Im}(S_k) \cdot 2r_k^h \left[\cos(h\theta_k) \operatorname{Im}(X_{k,t}) + \sin(h\theta_k) \operatorname{Re}(X_{k,t})\right] \end{aligned} \tag{50}$$

Proof. Express CCPs of S_k, $X_{k,t}$, D_k, and D_k^h in terms of their real and imaginary components, i.e.:

$$\begin{aligned} S_k &= \operatorname{Re}(S_k) \pm i \operatorname{Im}(S_k) \\ X_{k,t} &= \operatorname{Re}(X_{k,t}) \pm i \operatorname{Im}(X_{k,t}) \\ D_k &= r_k \exp(\pm i\theta_k) \\ D_k^h &= r_k^h \exp(\pm ih\theta_k) \\ &= r_k^h \left[\cos(h\theta_k) \pm i \sin(h\theta_k)\right] \end{aligned} \tag{51}$$

where $r = |D_k|$ and $\theta_k = \cos^{-1}\left[\operatorname{Re}(D_k) / |D_k|\right]$. Then:

$$\begin{aligned} & S_k D_k^h X_{k,t} + \overline{S_k D_k^h X_{k,t}} \\ = & \left[\operatorname{Re}(S_k) + i \operatorname{Im}(S_k)\right] \cdot r_k^h \left[\cos(h\theta_k) + i \sin(h\theta_k)\right] \cdot \left[\operatorname{Re}(X_{k,t}) + i \operatorname{Im}(X_{k,t})\right] \\ & + \left[\operatorname{Re}(S_k) - i \operatorname{Im}(S_k)\right] \cdot r_k^h \left[\cos(h\theta_k) - i \sin(h\theta_k)\right] \cdot \left[\operatorname{Re}(X_{k,t}) - i \operatorname{Im}(X_{k,t})\right] \end{aligned} \tag{52}$$

There are eight terms in each expansion of the last two lines, i.e.:

$$\begin{aligned} & r_k^h \cdot \left[\operatorname{Re}(S_k) + i \operatorname{Im}(S_k)\right] \cdot \left[\cos(h\theta_k) + i \sin(h\theta_k)\right] \cdot \left[\operatorname{Re}(X_{k,t}) + i \operatorname{Im}(X_{k,t})\right] \\ = & r_k^h \cos(h\theta_k) \operatorname{Re}(S_k) \operatorname{Re}(X_{k,t}) - r_k^h \sin(h\theta_k) \operatorname{Re}(S_k) \operatorname{Im}(X_{k,t}) \\ & -r_k^h \cos(h\theta_k) \operatorname{Im}(S_k) \operatorname{Im}(X_{k,t}) - r_k^h \sin(h\theta_k) \operatorname{Im}(S_k) \operatorname{Re}(X_{k,t}) \\ & +ir_k^h \sin(h\theta_k) \operatorname{Re}(S_k) \operatorname{Re}(X_{k,t}) + ir_k^h \cos(h\theta_k) \operatorname{Re}(S_k) \operatorname{Im}(X_{k,t}) \\ & -ir_k^h \sin(h\theta_k) \operatorname{Im}(S_k) \operatorname{Im}(X_{k,t}) + ir_k^h \cos(h\theta_k) \operatorname{Im}(S_k) \operatorname{Re}(X_{k,t}) \end{aligned} \tag{53}$$

and:

$$\begin{aligned} & r_k^h \cdot \left[\operatorname{Re}(S_k) - i \operatorname{Im}(S_k)\right] \cdot \left[\cos(h\theta_k) - i \sin(h\theta_k)\right] \cdot \left[\operatorname{Re}(X_{k,t}) - i \operatorname{Im}(X_{k,t})\right] \\ = & r_k^h \cos(h\theta_k) \operatorname{Re}(S_k) \operatorname{Re}(X_{k,t}) - r_k^h \sin(h\theta_k) \operatorname{Re}(S_k) \operatorname{Im}(X_{k,t}) \\ & -r_k^h \cos(h\theta_k) \operatorname{Im}(S_k) \operatorname{Im}(X_{k,t}) - r_k^h \sin(h\theta_k) \operatorname{Im}(S_k) \operatorname{Re}(X_{k,t}) \\ & -ir_k^h \sin(h\theta_k) \operatorname{Re}(S_k) \operatorname{Re}(X_{k,t}) - ir_k^h \cos(h\theta_k) \operatorname{Re}(S_k) \operatorname{Im}(X_{k,t}) \\ & +ir_k^h \sin(h\theta_k) \operatorname{Im}(S_k) \operatorname{Im}(X_{k,t}) - ir_k^h \cos(h\theta_k) \operatorname{Im}(S_k) \operatorname{Re}(X_{k,t}) \end{aligned} \tag{54}$$

Summing the two sets of results gives zero for the imaginary terms, and the real terms obtain the expression in equation 50 The result for $D_k^h X_{k,t} + \overline{D_k^h X_{k,t}}$ could be obtained with a similar expansion and cancellation, but an easier alternative is to substitute $1 = 1 + 0i$ for S_k in equation 50, hence $\operatorname{Re}(1+0i) = 1$ and $\operatorname{Im}(1+0i) = 0$, so:

$$D_k^h X_{k,t} + \overline{D_k^h X_{k,t}} = 2r_k^h \cos(h\theta_k) \operatorname{Re}(X_{k,t}) - 2r_k^h \sin(h\theta_k) \operatorname{Im}(X_{k,t}) \tag{55}$$

■

Proposition 9 *For CCP eigenvalues, the associated FEV and ergodic variance components* $[\Omega_y(h)]_{k\&k+1}$ *and* $[\Omega_y(\infty)]_{k\&k+1}$ *may respectively be expressed as:*

$$[\Omega_y(h)]_{k\&k+1} = 2\operatorname{Re}\left(\Omega_{E_X,k,k}\frac{1-|D_k|^{2h}}{1-|D_k|^2}S_k S_k^\dagger\right) + 2\operatorname{Re}\left(\Omega_{E_X,k,k+1}\frac{1-D_k^{2h}}{1-D_k^2}S_k\overline{S_k^\dagger}\right)$$

and:

$$[\Omega_y(\infty)]_{k\&k+1} = 2\operatorname{Re}\left(\Omega_{E_X,k,k}\frac{1}{1-|D_k|^2}S_k S_k^\dagger\right) + 2\operatorname{Re}\left(\Omega_{E_X,k,k+1}\frac{1}{1-D_k^2}S_k\overline{S_k^\dagger}\right) \tag{56}$$

Proof. For distinct eigenvalues where (D_k, D_{k+1}) is either a pair of real eigenvalues or CCP eigenvalues, the FEV of the forecast component $S_k D_k^h X_{k,t} + S_{k+1} D_{k+1}^h X_{k+1,t}$ is:

$$\begin{aligned}
[\Omega_y(h)]_{[k]} &= [S_k, S_{k+1}] \begin{bmatrix} \Omega_{E_X,k,k}\frac{1-\left(D_k\overline{D_k}\right)^h}{1-D_k\overline{D_k}} & \Omega_{E_X,k,k+1}\frac{1-\left(D_k\overline{D_{k+1}}\right)^h}{1-D_k\overline{D_{k+1}}} \\ \Omega_{E_X,k,k+1}\frac{1-\left(D_k\overline{D_{k+1}}\right)^h}{1-D_k\overline{D_{k+1}}} & \Omega_{E_X,k+1,k+1}\frac{1-\left(D_{k+1}\overline{D_{k+1}}\right)^h}{1-D_{k+1}\overline{D_{k+1}}} \end{bmatrix} \begin{bmatrix} S_k^\dagger \\ S_{k+1}^\dagger \end{bmatrix} \\
&= [S_k, S_{k+1}] [\Omega_{X,k:k+1}(h)] [S_k, S_{k+1}]^\dagger
\end{aligned} \tag{57}$$

where $\Omega_{X,k:k+1}(h)$ denotes the 2×2 block of $\Omega_X(h)$ associated with (D_k, D_{k+1}).

For CCP eigenvalues $(D_k, D_{k+1}) = \left(D_k, \overline{D_k}\right)$:

$$\begin{aligned}
[\Omega_y(h)]_{k\&k+1} &= \left[S_k, \overline{S_k}\right] \begin{bmatrix} \Omega_{E_X,k,k}\frac{1-|D_k|^{2h}}{1-|D_k|^2} & \Omega_{E_X,k,k+1}\frac{1-D_k^{2h}}{1-D_k^2} \\ \overline{\Omega_{E_X,k,k+1}\frac{1-D_k^{2h}}{1-D_k^2}} & \Omega_{E_X,k,k}\frac{1-|D_k|^{2h}}{1-|D_k|^2} \end{bmatrix} \begin{bmatrix} S_k^\dagger \\ \overline{S_k^\dagger} \end{bmatrix} \\
&= \Omega_{E_X,k,k}\frac{1-|D_k|^{2h}}{1-|D_k|^2}S_k S_k^\dagger + \Omega_{E_X,k,k}\frac{1-|D_k|^{2h}}{1-|D_k|^2}\overline{S_k S_k^\dagger} \\
&\quad + \Omega_{E_X,k,k+1}\frac{1-D_k^{2h}}{1-D_k^2}S_k\overline{S_k^\dagger} + \overline{\Omega_{E_X,k,k+1}\frac{1-D_k^{2h}}{1-D_k^2}}\,\overline{S_k}S_k^\dagger \\
&= 2\operatorname{Re}\left(\Omega_{E_X,k,k}\frac{1-|D_k|^{2h}}{1-|D_k|^2}S_k S_k^\dagger\right) + 2\operatorname{Re}\left(\Omega_{E_X,k,k+1}\frac{1-D_k^{2h}}{1-D_k^2}S_k\overline{S_k^\dagger}\right)
\end{aligned} \tag{58}$$

The ergodic variance is $\lim_{h\to\infty}[\Omega_y(h)]_{k\&k+1}$, which is the result provided in the proposition.

■

7 EVAR components with repeated eigenvalues

This section provides details related to the discussion in section 3.3 that a 2×2 Jordan block is required in the eigenvalue matrix when a repeated pair of eigenvalues is imposed on a VAR. The related analysis proceeds analogous to the distinct eigenvalue with respect to the processes, components, and their forecasts, but based on the 2×2 Jordan block. Unlike the real AR2 forms in section 6 of this online appendix, the forecasts do have the convenience of closed-form solutions based on scalar powers, although the derivations required to obtain them is more involved than the elementary AR1 case.

Proposition 10 *For a repeated eigenvalue pair, the process for* $X_{[k],t}$*:*

$$X_{[k],t} = D_{[k]} X_{[k],t-1} + E_{X,[k],t} \tag{59}$$

where:

$$X_{[k],t} = \begin{bmatrix} X_{k,t} \\ X_{k+1,t} \end{bmatrix} \; ; \; D_{[k]} = \begin{bmatrix} D_k & 1 \\ 0 & D_k \end{bmatrix} \; ; \; X_{[k],t-1} = \begin{bmatrix} X_{k,t-1} \\ X_{k+1,t-1} \end{bmatrix} \; ; \; E_{X,[k],t} = \begin{bmatrix} E_{X,k,t} \\ E_{X,k+1,t} \end{bmatrix} \tag{60}$$

has closed-form solutions for forecasts of $X_{[k],t}$*, forecasts of* $[S_k, S_{k+1}] X_{[k],t}$*, FEVs* $[\Omega_y(h)]_{[k]}$*, and the ergodic variance* $[\Omega_y(\infty)]_{[k]}$*.*

Proof. From Hamilton (1994) p. 19, a Jordan block containing a repeated eigenvalue pair has powers:

$$\begin{bmatrix} D_1 & 1 \\ 0 & D_1 \end{bmatrix}^h = \begin{bmatrix} D_1^h & hD_1^{h-1} \\ 0 & D_1^h \end{bmatrix} \tag{61}$$

Using equation 61, closed-form forecasts of $[X_{1,t}, X_{2,t}]'$ associated with the Jordan block are:

$$\begin{aligned} \mathbb{E}_t \begin{bmatrix} X_{1,t+h} \\ X_{2,t+h} \end{bmatrix} &= \begin{bmatrix} D_1 & 1 \\ 0 & D_1 \end{bmatrix}^h \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} \\ &= \begin{bmatrix} D_1^h & hD_1^{h-1} \\ 0 & D_1^h \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} \\ &= \begin{bmatrix} D_1^h X_{1,t} + hD_1^{h-1} X_{2,t} \\ D_1^h X_{2,t} \end{bmatrix} \end{aligned} \tag{62}$$

and closed-form forecasts of the EVAR component is $[S_k, S_{k+1}] [X_{k,t}, X_{k+1,t}]'$ are:

$$\begin{aligned} [S_1, S_2] \begin{bmatrix} D_1 & 1 \\ 0 & D_1 \end{bmatrix}^h \begin{bmatrix} X_{k,t} \\ X_{k+1,t} \end{bmatrix} &= [S_1, S_2] \left[\begin{bmatrix} D_1^h X_{1,t} + hD_1^{h-1} X_{2,t} \\ D_1^h X_{2,t} \end{bmatrix} \right] \\ &= S_1 D_1^h X_{1,t} + S_1 \cdot hD_1^{h-1} X_{2,t} + S_2 D_1^h X_{2,t} \end{aligned} \tag{63}$$

Closed-form FEVs are $[\Omega_y(h)]_{[k]} = [S_k, S_{k+1}] [\Omega_X(h)]_{[k]} [S_k, S_{k+1}]'$ where

$$\begin{aligned} [\Omega_y(h)]_{[k]} &= [S_1, S_2] \left[\sum_{n=0}^{h-1} D_{[k]}^n \Omega_{E_X,[k]} \left(D_{[k]}' \right)^n \right] [S_1, S_2]' \\ &= [S_1, S_2] \left[\sum_{n=0}^{h-1} \begin{bmatrix} D_1 & 1 \\ 0 & D_1 \end{bmatrix}^n \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 1 & D_1 \end{bmatrix}^n \right] \begin{bmatrix} S_1' \\ S_2' \end{bmatrix} \end{aligned} \tag{64}$$

where $\Omega_{E_X,[k]}$ is presented in generic form for notational convenience in what follows.

Each matrix $D_{[k]}^n \Omega_{E_X,[k]} \left(D_{[k]}' \right)^n$ in $[\Omega_X(h)]_{[k]} = \sum_{n=0}^{h-1} D_{[k]}^n \Omega_{E_X,[k]} \left(D_{[k]}' \right)^n$ is:

$$D_{[k]}^n \Omega_{E_X} \left(D_{[k]}^{\dagger} \right)^n = \begin{bmatrix} D_k^n & hD_k^{n-1} \\ 0 & D_k^n \end{bmatrix} \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \begin{bmatrix} D_k^n & 0 \\ hD_k^{n-1} & D_k^n \end{bmatrix}$$

The $(1,1)$ element is:

$$\begin{aligned}
& \left[D^n \Omega_{E_X} \left(D^\dagger\right)^n\right]_{11} \\
= & \left[D_1^n, nD_1^{n-1}\right] \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \begin{bmatrix} D_1^n \\ nD_1^{n-1} \end{bmatrix} \\
= & \Omega_{11} \cdot D_1^{2n} + \Omega_{12} \cdot 2nD_1^{2n-1} + \Omega_{22} \cdot n^2 D_1^{2n-2}
\end{aligned} \tag{65}$$

the $(1,2)$ element of is:

$$\begin{aligned}
& \left[D^n \Omega_{E_X} \left(D^\dagger\right)^n\right]_{12} \\
= & \left[D_1^n, hD_1^{n-1}\right] \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \begin{bmatrix} 0 \\ D_1^n \end{bmatrix} \\
= & \Omega_{12} \cdot D_1^{2n} + \Omega_{22} \cdot hD_1^{2n-1}
\end{aligned} \tag{66}$$

and the $(2,2)$ element is:

$$\begin{aligned}
& \left[D^n \Omega_{E_X} \left(D^\dagger\right)^n\right]_{12} \\
= & \left[0, D_1^n\right] \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \begin{bmatrix} 0 \\ D_1^n \end{bmatrix} \\
= & \Omega_{22} \cdot D_1^{2n}
\end{aligned} \tag{67}$$

so therefore:

$$D_{[k]}^n \Omega_{E_X} \left(D_{[k]}^\dagger\right)^n = \begin{bmatrix} \Omega_{11} D_k^{2n} + 2n\Omega_{12} D_k^{2n-1} + n^2 \Omega_{22} D_k^{2n-2} & \Omega_{12} D_k^{2n} + n\Omega_{22} D_k^{2n-1} \\ \Omega_{12} D_k^{2n} + n\Omega_{22} D_k^{2n-1} & \Omega_{22} D_k^{2n} \end{bmatrix} \tag{68}$$

The summation $\sum_{n=0}^{h-1} D_{[k]}^n \Omega_{E_X,[k]} \left(D'_{[k]}\right)^n$ therefore requires the summations $\sum_{n=0}^{h-1} D_k^{2n}$, $\sum_{n=0}^{h-1} nD_1^{2n-1}$, and $\sum_{n=0}^{h-1} n^2 D^{2n-2}$. These have closed-form expressions, as detailed in the following proposition and its proof below, and so $\left[\Omega_y(h)\right]_{[k]}$ has a closed-form solution. ■

Proposition 11 *The summations* $\sum_{n=0}^{h-1} D_k^{2n}$, $\sum_{n=0}^{h-1} nD_1^{2n-1}$, *and* $\sum_{n=0}^{h-1} n^2 D^{2n-2}$ *and their respective infinite limits* $\sum_{n=0}^{\infty} (\cdot)$ *have closed-form solutions:*

$$\begin{array}{rcl}
\sum_{n=0}^{h-1} D_1^{2n} = \frac{1-D_1^{2h}}{1-D_1^2} & ; & \lim_{n\to\infty} \sum_{n=0}^{h-1} D_1^{2n} = \frac{1}{1-D_1^2} \\
\sum_{n=0}^{h-1} nD_1^{2n-1} = \frac{D_1 - hD_1^{2h-1} + (h-1)D_1^{2h+1}}{\left(1-D_1^2\right)^2} & ; & \lim_{n\to\infty} \sum_{n=0}^{h-1} nD^{2n-1} = \frac{D_1}{\left(1-D_1^2\right)^2} \\
\sum_{n=0}^{h-1} n^2 D^{2n-2} = \frac{1}{D_1^2} \cdot \sum_{n=0}^{h-1} n^2 x^n \Big|_{x=D_1^2} & ; & \lim_{n\to\infty} \sum_{n=0}^{h-1} n^2 D^{2n-2} = \frac{1+D_1^2}{\left(1-D_1^2\right)^3}
\end{array} \tag{69}$$

Proof. $\sum_{n=0}^{h-1} x^n$ in closed-form is obtained as:

$$\begin{aligned}
(1-x)\sum_{n=0}^{h-1} x^n &= \sum_{n=0}^{h-1} x^n - \sum_{n=0}^{h-1} x^{n+1} \\
&= 1 + \sum_{n=1}^{h-1} x^n - \left[\sum_{n=0}^{h-2} x^{n+1}\right] - x^h \\
&= 1 - x^h + \sum_{n=1}^{h-1} x^n - \sum_{n=1}^{h-1} x^n \\
&= 1 - x^h \\
\sum_{n=0}^{h-1} x^n &= \frac{1-x^h}{1-x}
\end{aligned} \tag{70}$$

and so:

$$\begin{aligned}\sum_{n=0}^{h-1} D_1^{2n} &= \frac{1-D_1^{2h}}{1-D_1^2}\\ \lim_{n\to\infty}\sum_{n=0}^{h-1} D_1^{2n} &= \frac{1}{1-D_1^2}\end{aligned}$$

$\sum_{n=0}^{h-1} nx^n$ in closed-form is obtained as:

$$\begin{aligned}x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{h-1} x^n &= x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-x^h}{1-x}\right)\\ x\left(1-x\right)^2\sum_{n=0}^{h-1} nx^{n-1} &= -xhx^{h-1}\left(1-x\right)+x\left(1-x^h\right)\\ \sum_{n=0}^{h-1} nx^n &= \frac{x-hx^h+\left(h-1\right)x^{h+1}}{\left(1-x\right)^2}\end{aligned}$$

and so:

$$\begin{aligned}\sum_{n=0}^{h-1} nD_1^{2n-1} &= \frac{1}{D_1}\cdot\sum_{n=0}^{h-1} n\left(D_1^2\right)^n\\ &= \frac{1}{D_1}\cdot\frac{D_1^2-hD_1^{2h}+\left(h-1\right)D_1^{2h+2}}{\left(1-D_1^2\right)^2}\\ &= \frac{D_1-hD_1^{2h-1}+\left(h-1\right)D_1^{2h+1}}{\left(1-D_1^2\right)^2}\\ \lim_{n\to\infty}\sum_{n=0}^{h-1} nD^{2n-1} &= \frac{D_1}{\left(1-D_1^2\right)^2} \qquad (71)\end{aligned}$$

$\sum_{n=0}^{h-1} n^2x^n$ in closed-form is obtained as:

$$\begin{aligned}x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{h-1} nx^n &= x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-hx^h-x^{h+1}+hx^{h+1}}{\left(1-x\right)^2}\right)\\ x\left(1-x\right)^4\sum_{n=0}^{h-1} n^2x^{n-1} &= x\left(1-x^h-h^2x^{h-1}+h^2x^h\right)\left(1-x\right)^2\\ &\quad +2x\left(x-hx^h-x^{h+1}+hx^{h+1}\right)\left(1-x\right)\\ \left(1-x\right)^3\sum_{n=0}^{h-1} n^2x^n &= x\left(1-x^h-h^2x^{h-1}+h^2x^h\right)\left(1-x\right)+2x\left(x-hx^h-x^{h+1}+hx^{h+1}\right)\\ \sum_{n=0}^{h-1} n^2x^n &= \frac{x+x^2-h^2x^h+\left(2h^2-2h-1\right)x^{h+1}-\left(h-1\right)^2x^{h+2}}{\left(1-x\right)^3} \qquad (72)\end{aligned}$$

and so:

$$\sum_{n=0}^{h-1} n^2D^{2n-2} = \frac{1}{D_1^2}\cdot\sum_{n=0}^{h-1} n^2x^n\bigg|_{x=D_1^2} \qquad (73)$$

where $\sum_{n=0}^{h-1} n^2 x^n \Big|_{x=D_1^2}$ denotes substituting D_1^2 for x in the closed-form solution on the right-hand side of equation 72, and:

$$\begin{aligned} \lim_{n\to\infty} \sum_{n=0}^{h-1} n^2 D^{2n-2} &= \frac{1}{D_1^2} \cdot \frac{D_1^2 + D_1^4}{\left(1 - D_1^2\right)^3} \\ &= \frac{1 + D_1^2}{\left(1 - D_1^2\right)^3} \end{aligned} \qquad (74)$$

■

8 Complete set of ergodic variance results

This section provides additional ergodic variance results related to the second application in section 4 of the main text. These include the ergodic variances of the individual EVAR components, and also the cumulative contributions of the EVAR components.

The left side of table A1 contains the ergodic variance results for all of the individual EVAR components. These are $[\Omega_y(\infty)]_k = S_k \Omega_{X,k,k} S_k^\dagger = \Omega_{X,k,k} S_k S_k'$ for the real eigenvalues, and the top $N \times N$ block is the example for $[\Omega_y(\infty)]_1$ contained in the main text, i.e.:

$$[\Omega_y(\infty)]_1 = \begin{bmatrix} 1.92 & 2.40 & 3.72 \\ 2.40 & 3.01 & 4.65 \\ 3.72 & 4.65 & 7.19 \end{bmatrix} \qquad (75)$$

For the CCP eigenvalues (D_3, D_4):

$$\begin{aligned} [\Omega_y(\infty)]_{3\&4} &= [S_3, S_4] \begin{bmatrix} \Omega_{X,3,3}(\infty) & \Omega_{X,3,4}(\infty) \\ \overline{\Omega_{X,3,4}(\infty)} & \Omega_{X,4,4}(\infty) \end{bmatrix} [S_3, S_4]^\dagger \\ &= [S_3, S_4] [\Omega_{X,3:4}(\infty)] [S_3, S_4]^\dagger \end{aligned} \qquad (76)$$

where $\Omega_{X,3:4}(\infty)$ denotes the 2×2 block of $\Omega_X(\infty)$ associated with (D_3, D_4).

The individual EVAR component results do not account for covariances between the different EVAR components, i.e. the off-diagonal elements of $\Omega_X(\infty)$, such as $\Omega_{X,1,2}(\infty)$ or $\Omega_{X,1,3:4}(\infty)$. Therefore, summing the individual EVAR components results, i.e. $[\Omega_y]_1 + [\Omega_y]_2 + [\Omega_y]_{3\&4} + [\Omega_y]_5 + [\Omega_y]_6$, will not give $\Omega_y(\infty)$ from the main text, i.e.:

$$\Omega_y(\infty) = \begin{bmatrix} 2.18 & 1.19 & 1.60 \\ 1.19 & 7.55 & 5.81 \\ 1.60 & 5.81 & 8.42 \end{bmatrix} \qquad (77)$$

The cumulative expression that accounts for covariances is:

$$[\Omega_y(\infty)]_{1:k} = [S_1, \ldots, S_k] [\Omega_{X,1:k}(\infty)] [S_1, \ldots, S_k]^\dagger \qquad (78)$$

where $\Omega_{X,1:k}$ denotes the $k \times k$ sub-matrix from the first row and column of Ω_X to row k and column k. The cumulation of contributions to the ergodic variance $\Omega_y(\infty)$ is shown on the right-hand side of table A1. $\Omega_y(\infty)$ is the full cumulation of ergodic variance components, i.e. the last matrix on the right-hand side, which is also evident from:

$$\begin{aligned} \Omega_y(\infty) &= [\Omega_y(\infty)]_{1:6} \\ &= [S_1, \ldots, S_6] [\Omega_{X,1:6}(\infty)] [S_1, \ldots, S_6]^\dagger \\ &= S \Omega_X(\infty) S^\dagger \end{aligned} \qquad (79)$$

Table A1: Ergodic variance components and cumulation

		$[\Omega_y]_k$			$[\Omega_y]_{1:k}$		
		u	π	r	u	π	r
	u	1.92	2.40	3.72	1.92	2.40	3.72
$k = 1$	π	2.40	3.01	4.65	2.40	3.01	4.65
	r	3.72	4.65	7.19	3.72	4.65	7.19
	u	1.01	0.91	–1.73	3.88	4.34	2.09
$k = 2$	π	0.91	0.83	–1.56	4.34	4.91	2.90
	r	–1.73	–1.56	2.97	2.09	2.90	7.00
	u	3.94	–1.20	–2.53	2.20	1.31	1.54
$k = 3\&4$	π	–1.20	0.83	0.68	1.31	8.21	5.46
	r	–2.53	0.68	1.64	1.54	5.46	8.30
	u	0.00	0.02	–0.01	2.18	1.19	1.60
$k = 5$	π	0.02	0.09	–0.07	1.19	7.57	5.76
	r	–0.01	–0.07	0.06	1.60	5.76	8.23
	u	0.00	0.00	–0.00	2.18	1.19	1.60
$k = 6$	π	0.00	0.00	–0.00	1.19	7.55	5.81
	r	–0.00	–0.00	0.02	1.60	5.81	8.42

A point of note from table A1 is that the components 5 and 6 are both immaterial, both as individual components $[\Omega_y]_5$ and $[\Omega_y]_6$, and their contributions to Ω_y. That is the cumulation of the first four components $[\Omega_y]_{1:4,1:4} \simeq \Omega_y$. The immateriality of components 5 and 6 may also be seen from the last two panels of figure 1.

9 VAR estimates with a zero eigenvalue constraint

This section contains the results for the first example of the third application in section 4 of the main text, i.e. estimating the VAR subject to the eigenvalue constraint $D_6 = 0$.

Table A2: VAR coefficient and covariance matrix estimates

	coefficients						covariances		
	β_1			β_2			Ω_ε		
β_u	1.37	–0.02	0.03	–0.46	0.04	–0.01	0.11	–0.02	–0.12
β_π	–0.34	1.12	0.15	0.30	–0.22	–0.08	–0.02	0.67	0.20
β_r	–0.65	–0.09	0.77	0.66	0.18	0.13	–0.12	0.20	0.87

Table A3: VAR eigensystem parameters

k	1	2	3	4	5	6
D_k	0.97	0.78	0.72+0.13i	0.72–0.13i	0.07	0.00
$S_{k,u}$	0.52	–0.34	–1.68+0.23i	–1.68–0.23i	–0.10	–0.07
$S_{k,\pi}$	0.63	–0.60	1.14–0.78i	1.14+0.78i	–0.67	–0.46
$S_{k,r}$	1.00	1.00	1.00	1.00	1.00	1.00

10 VAR estimates with a repeated eigenvalue constraint

This section contains the results for the second example of the third application in section 4 of the main text, i.e. estimating the VAR subject to the eigenvalue constraint $D_3 = D_4$.

Table A4: VAR coefficient and covariance matrix estimates

	coefficients						covariances		
	β_1			β_2			Ω_ε		
β_u	1.35	–0.02	0.03	–0.43	0.04	–0.00	0.11	–0.02	–0.12
β_π	–0.34	1.14	0.12	0.29	–0.24	–0.05	–0.02	0.67	0.20
β_r	–0.63	–0.08	0.74	0.62	0.16	0.16	–0.12	0.20	0.87

Table A5: VAR eigensystem parameters

k	1	2	3	4	5	6
D_k	0.97	0.735	0.725	0.725	0.20	–0.14
$S_{k,u}$	0.56	–1.33	–1.44	8.14	–0.18	–0.02
$S_{k,\pi}$	0.63	0.42	0.59	–15.08	–1.22	–0.16
$S_{k,r}$	1	1	1	1	1	1

11 VAR estimates with a median-unbiased eigenvalue constraint

This section contains the results for estimating the VAR with D_1 set to the median-unbiased estimate of D_1 from the initial VAR, i.e. $\mathrm{MU}(D_1) = 0.9920$.

Table A6: VAR coefficient and covariance matrix estimates

	coefficients						covariances		
	β_1			β_2			Ω_ε		
β_u	1.37	–0.02	0.03	–0.46	0.04	–0.01	0.11	–0.02	–0.12
β_π	–0.36	1.15	0.12	0.34	–0.25	–0.03	–0.02	0.67	0.20
β_r	–0.68	–0.06	0.74	0.70	0.15	0.18	–0.12	0.20	0.87

Table A7: VAR eigensystem parameters

k	1	2	3	4	5	6
D_k	0.99	0.75	0.72+0.15i	0.72–0.15i	0.22	-0.15
$S_{k,u}$	0.43	–0.55	–1.71+0.18i	–1.71–0.18i	–0.22	–0.02
$S_{k,\pi}$	0.75	–0.55	1.33–0.86i	1.33+0.86i	–1.23	-0.12
$S_{k,r}$	1	1	1	1	1	1

12 VAR estimates with a unit root eigenvalue constraint

This section contains the results for estimating the VAR with the imposed constraint of $D_1 = 1$.

Table A8: VAR coefficient and covariance matrix estimates

	coefficients						covariances		
	β_1			β_2			Ω_ε		
β_u	1.37	–0.02	0.03	–0.46	0.04	–0.01	0.11	–0.02	-0.12
β_π	–0.36	1.15	0.12	0.34	–0.25	–0.03	–0.02	0.67	0.21
β_r	–0.68	–0.06	0.74	0.70	0.15	0.18	–0.12	0.21	0.88

Table A9: VAR eigensystem parameters

k	1	2	3	4	5	6
D_k	1.00	0.75	0.72+0.15i	0.72–0.15i	0.22	-0.15
$S_{k,u}$	0.40	–0.55	–1.71+0.17i	–1.71–0.17i	–0.22	-0.02
$S_{k,\pi}$	0.78	–0.55	1.34–0.85i	1.34+0.85i	–1.23	-0.12
$S_{k,r}$	1	1	1	1	1	1

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