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Trend-cycle-seasonal interactions: identification and estimation

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Economists typically use seasonally adjusted data in which the assumption is imposed that seasonality is uncorrelated with trend and cycle. The importance of this assumption has been highlighted by the Great Recession. The paper examines an unobserved components model that permits non-zero correlations between seasonal and non-seasonal shocks. Identification conditions for estimation of the parameters are discussed from the perspectives of both analytical and simulation results. Applications to UK household consumption expenditures and US employment reject the zero correlation restrictions and also show that the correlation assumptions imposed have important implications about the evolution of the trend and cycle in the post-Great Recession period.

Keywords

Trend-cycle-seasonal decomposition, unobserved components, seasonal adjustment, employment, Great Recession

JEL Classification

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Trend-cycle-seasonal interactions: identification and estimation*

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1 Introduction

Economic time series are typically analysed in *seasonally adjusted* form. That is, (estimated) seasonality is removed prior to undertaking substantive analysis of economic questions. Seasonal adjustment is based on the unobserved component approach, of which the key assumption is that the components (typically trend, cycle and seasonal) are mutually uncorrelated. However, a growing recent literature strongly suggests that the trend and cycle can be correlated; see Morley, Nelson and Zivot (2003), MNZ hereafter, Dungey, Jacobs, Tian and van Norden (2015) and others. While this has important implications for economic analyses that employ detrended data, the consequences of the uncorrelated assumption for seasonality are much more pervasive. Building on MNZ and the literature that indicates, on both economic and statistical and economic grounds, that cyclical and seasonal components may be correlated (including Cecchetti and Kashyap, 1996, Matas-Mir and Osborn, 2004), this paper extends the trend-cycle decomposition literature for economic time series to include the seasonal component.

The behaviour of series in the immediate aftermath of the Great Recession has provided an impetus for economists to examine seasonality and its treatment through seasonal adjustment. The zero correlation assumption is fundamental to seasonal adjustment because the resulting seasonally adjusted series can then be analysed without concern about the 'noise' of seasonality. However, Wright (2013) concludes that official seasonal adjustment distorted US employment data during the downturn of the Great Recession. Further, in commenting on Wright's (2013) paper, Stock (2013) questions the component independence assumption embedded in seasonal adjustment and advocates more work on the "important but neglected topic" of seasonality. In practice, experts in seasonal adjustment within the US Bureau of the Census and other official statistical agencies recognise that extraction of the seasonal component is particularly difficult during recessions (Evans and Tiller, 2013, Lytras and Bell, 2013) and that special treatment may be required. More fundamentally, however, these considerations question the assumption that seasonality evolves independently of the other characteristics of economic time series.

Following the tradition that dates back to at least Grether and Nerlove (1970) and Engle (1978), and also underlines the *structural time series* approach used by Harvey (1990) and Durbin and Koopman (2012), our approach is to consider an unobserved component (UC) model in which the individual time series components are specified as being both economically meaningful and often employed in empirical analyses. However, rather than maintaining the uncorrelated components assumption, we follow MNZ and allow non-zero correlation between the innovations to the components in order to investigate the implications for quarterly time series. More specifically, we investigate whether the underlying parameters are identified when the zero correlation assumption is relaxed, and examine the practical implications for the trend and cycle components of allowing nonzero correlations for the key macroeconomic time series of UK household consumption and US non-farm payroll employment.

Our analysis is based on the UC trend-cycle model employed by MNZ and widely used by macroeconomists because it captures the key characteristics believed to be typical of important 'real-world' series. To this we add a stochastic seasonal component, also modelled in typical fashion, and then examine whether the parameters are identified when a general cross-correlation structure is permitted. In related work, McElroy and Maravall (2014) examine identification from a more statistical perspective, but the model they consider does not include a stationary cyclical component of the form often posited by macroeconomists. Indeed, as shown by MNZ, such a cyclical component, represented by a model with AR order $p \geq 2$, is required for the two components of a trend-cycle model to be identified in the presence of cross-correlated innovations. Our analysis can be seen as an extension of MNZ that views seasonality as an integral part of the dynamic evolution of the macroeconomy.

We show that adding this seasonal component to the standard trend-cycle quarterly specification leads to hidden linear dependencies between the autocovariances of the model. Although the model apparently has sufficient nonzero autocovariances for estimation of all parameters, it fails to satisfy the rank condition. Consequently, the model is under identified, and additional restrictions are required for identification. Nevertheless, it is emphasised that the usual uncorrelated innovation assumption is not the only solution to the identification problem: only a single restriction is required and the over-identification assumptions of the uncorrelated model can be tested. Simulations illustrate the implications of estimation for both the unidentified and a correctly identified model.

The applications to UK household consumption and US non-farm payroll employment reject the conventional uncorrelated innovation assumption. However, echoing to some extent the findings of Wright (2013), we show that the correlation assumption imposed has substantial implications for the estimated trend and cycle components in the period after the Great Recession. For the case of US non-farm payroll employment, imposition of uncorrelated components implies a substantially deeper recession (interpreted as negative cycle values) than assuming a zero correlation for trend and seasonal innovations only or assuming perfect negative correlation for the trend-cycle innovations, the latter being the implicit assumption made in the Beveridge-Nelson trend-cycle decomposition (Beveridge and Nelson, 1981; Anderson, Low and Snyder, 2006). Indeed, the preferred statistical model for both series is a form of the *Single Source of Error* (SSE) model, where a common shock drives all components (Ord, Koehler and Snyder 1997; De Livera, Hyndman and Snyder 2011). However, the estimated trend and cycle properties for UK consumption are not plausible in economic terms.

The remainder of this paper is structured as follows. Section 2 presents the UC model we study with uncorrelated and correlated innovations. Section 3 and Section 4 discuss identification and simulation results, respectively. Section 5 presents empirical results for real UK household consumption and US employment, while Section 6 offers some concluding remarks.

2 The Model

As noted in the Introduction, a growing literature provides empirical evidence that the trend (permanent) and cycle (transient) components of economic time series are correlated. As discussed by Weber (2011), the economic rationale for such correlation can include real business cycle theories, nominal rigidities, hysteresis, policy responses to temporary shocks, and so on. Estimates of the correlation between the innovations of the trend and cycle for output or related series (such as employment) are negative and relatively close to -1; for example, MNZ, Sinclair (2010), Weber (2011), Dungey et al. (2015).

Due to the prevalent use of seasonally adjusted data, there is not a large existing literature concerning correlation of the seasonal with other components. Nevertheless, Barsky and Miron (1989) and Beaulieu, MacKie-Mason and Miron (1992) observe that seasonal and business cycles have common characteristics, while other studies find that seasonal patterns change with the stage of the business cycle (Canova and Ghysels 1994; Cecchetti and Kashyap 1996; Krane and Wascher 1999; Matas-Mir and Osborn 2004) and/or the trend (Koopman and Lee 2009). In particular, Cecchetti and Kashyap (1996) observe that seasonal cycles in production are less marked in business cycle booms, implying negative correlation between these components. As noted by Proietti (2006) negative correlations lead to higher weights on future observations in the Kalman smoother, resulting in relatively large revisions to filtered estimates; see also Dungey et al. (2015).

To reflect these findings, the model employed in our analysis is designed to be sufficiently general to capture potential correlations across component innovations, while also being of a form recognized by economists as capturing the essential features of macroeconomic time series.

2.1 Component specification

The UC model we consider is designed to be of a form that a macroeconomist might employ when taking account of seasonality alongside trend and cycle components in a quarterly time series. Therefore, the observed seasonal series $y_t, t = 1, 2, ...$ consists of a trend τ_t, a cycle c_t and a seasonal s_t component, with

$$y_t = \tau_t + c_t + s_t. \tag{1}$$

Each of these components has a natural interpretation. Following many previous studies, the trend and cycle components are given by

$$\tau_t = \tau_{t-1} + \beta + \eta_t \tag{2}$$

$$\phi(L)c_t = \varepsilon_t \tag{3}$$

where the p^{th} order autoregressive (AR) polynomial $\phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$ (L being the usual lag operator) has all roots strictly outside the unit circle. The random walk with drift specification for the trend, as in (2), is widely adopted in macroeconomics, while a pure AR, as in (3), is also typical for economic analysis. The AR order is often specified as $p = 2$, as in Clark (1987), Sinclair (2010) and the empirical application of MNZ; $p \geq 2$ allows the process for c_t to exhibit cyclic properties in the sense of a spectral peak at a business cycle frequency. However, $p > 2$ is rarely used in practice for quarterly seasonal macroeconomic time series, in order to keep the lags of the seasonal specification distinct from those of the cycle.

As widely applied in the UC literature, seasonality is represented in the so-called 'dummy variable' form,

$$S(L)s_t = \omega_t, \tag{4}$$

where (for quarterly data) $S(L) = 1 + L + L^2 + L^3$ is the annual summation operator for quarterly data; see Harvey (1989). The moving annual sum implied by $S(L)$ with stochastic ω_t permits seasonality to evolve over time, with the speed of this evolution dictated by the variance of the shock σ_s^2; $\sigma_s^2 = 0$ leads to deterministic seasonality that is constant over time. Wright (2013) estimates a special case of the model given by (1) to (4) with white noise cycle, $\phi(L) = 1$, and uncorrelated innovations for monthly US employment, using this to illustrate the statistical uncertainty surrounding seasonally adjusted values.

In may be noted that the components c_t and/or s_t are sometimes specified in a trigonometric form in the UC literature, with each then driven by two innovation processes which are assumed to be mutually uncorrelated. The use of such a specification would further complicate matters once correlation is allowed across components, and hence the simpler forms above are adopted in our analysis.

With τ_t, c_t and s_t as in (2)-(4), the innovation vector $\mathbf{v}_t = (\eta_t,\ \varepsilon_t,\ \omega_t)'$ has covariance matrix

$$\mathbf{Q} \equiv E[\mathbf{v}_t\mathbf{v}_t'] = \begin{bmatrix} \sigma_\tau^2 & \sigma_{\tau c} & \sigma_{\tau s} \\ \sigma_{\tau c} & \sigma_c^2 & \sigma_{cs} \\ \sigma_{\tau s} & \sigma_{cs} & \sigma_s^2 \end{bmatrix} \tag{5}$$

which is positive semi-definite. The standard assumption in the UC approach is uncorrelated innovations, namely the special case of diagonal $\mathbf{Q}$. However, following MNZ, recent interest in macroeconomics has focused around nonseasonal models which allow the trend-cycle correlation to be nonzero.

At the other extreme from diagonal $\mathbf{Q}$, the single source of error (SSE) model assumes the innovations that drive the components are perfectly correlated. Although the usual formulation of the SSE model, as in Ord, Koehler and Snyder (1997), specifies the measurement equation analogous to (1) with an idiosyncratic error and lagged rather than current component contributions, Anderson, Low and Snyder (2006) show that the perfectly correlated trend-cycle model employed by Beveridge and Nelson (1981) can be written in conventional SSE form.[1] For the model of (1), an SSE formulation has

$$\mathbf{v}_t = \begin{bmatrix} k_\tau \\ k_c \\ k_s \end{bmatrix} \upsilon_t, \tag{6}$$

with $\upsilon_t \sim i.i.d.(0,1)$ so that the disturbances of (2)-(4) are each written as a scalar multiple of a single shock. Hence the component disturbances are perfectly correlated with covariance matrix

$$\mathbf{Q} = \begin{bmatrix} k_\tau^2 & k_\tau k_c & k_\tau k_s \\ k_\tau k_c & k_c^2 & k_c k_s \\ k_\tau k_s & k_c k_s & k_s^2 \end{bmatrix} = \begin{bmatrix} \sigma_\tau^2 & \sigma_{\tau c} & \sigma_{\tau s} \\ \sigma_{\tau c} & \sigma_c^2 & \sigma_{cs} \\ \sigma_{\tau s} & \sigma_{cs} & \sigma_s^2 \end{bmatrix}. \tag{7}$$

Employing the trend-cycle model of (2) and (3), with the latter sometimes including a moving average, MNZ and a number of subsequent studies (including the ones cited in the introduction) discuss identification and empirically compare the implications for GDP of the correlation assumptions made in the traditional UC approach, the BN decomposition and with an estimated innovation correlation. However, these studies do not consider seasonality.

The properties of the model can be established through the univariate $ARMA$ representation. Due to the zero frequency unit root in (2) and the seasonal unit roots in (4), the process of (1) to (4) is stationary and invertible after annual differencing ($\Delta_4 = 1 - L^4$). The reduced form of the model is therefore

$$\phi(L)\Delta_4 y_t = \phi(L)S(L)\beta + \phi(L)S(L)\eta_t + \Delta_4\,\varepsilon_t + \phi(L)\Delta\omega_t. \tag{8}$$

Analogously to MNZ, and using standard results on the sum of the moving average terms on the right-hand side of (8), the reduced form $ARMA(p, q)$ specification for $\Delta_4 y_t$ is

$$\phi(L)\Delta_4 y_t = \delta + \theta(L)u_t, \tag{9}$$

where $\delta = \phi(L)S(L)\beta,\ \theta(L)$ is a q^{th} order polynomial in L with $q \leq \max(p + 3,\ 4)$ and u_t is a white noise disturbance with constant variance. Further details on the derivation of (9) can be found in the Technical Appendix, while the order q is discussed in the next section for the cases of interest to us.

3 Identification

Before attempting to estimate the UC model of the preceding section allowing a general correlation structure for the disturbances, it must first be established that the model is identified. As for any $ARMA(p, q)$ process, the autocovariances γ_k of $\Delta_4 y_t$ at lag k satisfy

$$\gamma_k = \phi_1\gamma_{k-1} + \ldots + \phi_p\gamma_{k-p}, \quad k > q \tag{10}$$

which identifies the AR coefficients of (3). Hence, the autocovariances of the MA component of (9) for $k = 0, ..., q$ must contain sufficient information to identify the parameters of (5). More specifically, defining $\sigma = [\sigma_\tau^2, \sigma_c^2, \sigma_s^2, \sigma_{\tau c}, \sigma_{\tau s}, \sigma_{cs}]'$ to contain the unique elements of the covariance matrix $\mathbf{Q}$ and also defining the vector of autocovariances $\gamma = [\gamma_0, \ldots, \gamma_q]'$, yields the system

$$\gamma = \mathbf{A}\sigma \tag{11}$$

where $\mathbf{A}$ is a $(q+1) \times (q+1)$ matrix of constants. Identification of the six parameters of (5) requires $\mathbf{A}$ to be of rank 6.

This section discusses this identification from a theoretical perspective, considering first the case where the cycle is white noise ($p = 0$), before turning to $p = 2$; the implications of an $AR(1)$ cycle are considered as a special case of the latter.

3.1 White noise cycle

With c_t in (1) white noise, the model considered is the quarterly analogue of the basic structural model examined by McElroy and Maravall (2014) for monthly data with, in their notation, $d = 1$. A simple 'counting' check shows that the model where the cycle is white noise ($p = 0$) cannot be identified, as $q < 5$ and the nonzero autocovariances are insufficient in number to identify the six parameters of $\mathbf{Q}$. Nevertheless, this case serves to illustrate some general features of identification which apply also in the more general AR cycle examined below.

For $p = 0$, the stochastic component on the right-hand side of (9) is

$$\begin{aligned} z_t &= S(L)\eta_t + \Delta_4 \varepsilon_t + \Delta\omega_t \\ &= \eta_t + ... + \eta_{t-3} + \varepsilon_t - \varepsilon_{t-4} + \omega_t - \omega_{t-1}. \end{aligned}$$

As shown in the Technical Appendix, except in the special case where $\sigma_c^2 = -(\sigma_{\tau c} + \sigma_{cs})$, z_t

is $MA(4)$ so that $\gamma_k = 0$ for $k > 4$ and the matrix $\mathbf{A}$ of (11) is

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 2 & 2 & 0 & 2 \\ 3 & 0 & -1 & 0 & -1 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{bmatrix}. \tag{12}$$

Although the model is not identified overall, nevertheless two variance parameters of $\mathbf{Q}$ can be obtained irrespective of any covariance assumptions. Specifically, the variances of the trend and seasonal innovations are given by

$$\begin{aligned} \sigma_\tau^2 &= 0.5\gamma_2 \\ \sigma_s^2 &= 2\gamma_2 - \gamma_1 - \gamma_3. \end{aligned}$$

This extends the trend-cycle case examined by MNZ, who note that the variance of the trend innovations can similarly be identified when the cycle is white noise, although the individual terms in $\sigma_c^2 + \sigma_{\tau c}$ cannot.[2]

Noting that σ_c^2 and $\sigma_{\tau c}$ never separately enter $\mathbf{A}$ of (12), it could be presumed that $\sigma_c^2 + \sigma_{\tau c}$ and the four other distinct parameters of $\mathbf{Q}$ will be identified. This is, however, not the case, since the rows of $\mathbf{A}$ are linearly dependent, with

$$\gamma_0 = -2\gamma_1 + 6\gamma_2 - 2\gamma_3 - 2\gamma_4.$$

Hence the system contains only four linearly independent equations, rather than five. Consequently it is not possible to identify either $\sigma_{\tau s}$ or σ_{cs} without further information. However, a single linear restriction on $\sigma_{\tau s}$ and/or σ_{cs} allows identification of σ_τ^2, σ_s^2, $(\sigma_c^2 + \sigma_{\tau c})$, $\sigma_{\tau s}$ and σ_{cs}, with a further restriction required to separate σ_c^2 and $\sigma_{\tau c}$.

This discussion underlines the importance for identification of the traditional uncorrelated disturbance of the UC model. It also shows the crucial role played by the uncorrelated innovation assumption in the illustrative model used by Wright (2013). Nevertheless, because there are four linearly independent nonzero γ_k and three unknown variances, uncorrelated innovations lead to the presence of an over-identifying restriction; hence some testing is possible. More explicitly, for the case under consideration, the single over-identifying restriction embodied in the uncorrelated innovation assumption could be interpreted as either $\sigma_{\tau s} = 0$ or $\sigma_{cs} = 0$, depending on the *a priori* views of the researcher. Consequently, although cycle parameters σ_c^2 and $\sigma_{\tau c}$ cannot be separated, the assumption implicit in seasonal adjustment that seasonality is uncorrelated with other components can be tested even when the cycle is white noise only.

3.2 AR(2) cycle

As noted in Section 2, and due to the stationary cycles it can imply, the case $p = 2$ is of great empirical interest to macroeconomists. However, it is not examined by McElroy and Maravall (2014). Note first that $p = 2$ implies $q \leq 5$ in (9) and, again unless $\sigma_c^2 = -(\sigma_{\tau c}+\sigma_{cs})$, q is equal to its upper limit (see the Technical Appendix). Consequently, the 'counting' requirement is fulfilled and the autocovariances of the right-hand side of (9) may potentially provide sufficient information to just identify the parameters of (5). Hence we check the rank condition.

For this $AR(2)$ case, the MA of the right-hand side of (8) is

$$\begin{aligned} z_t &= [1 + (1 - \phi_1)L + (1 - \phi_1 - \phi_2)L^2 + (1 - \phi_1 - \phi_2)L^3 - (\phi_1 + \phi_2)L^4 - \phi_2 L^5]\eta_t \\ &\quad +[1 - L^4]\varepsilon_t + [1 - (1 + \phi_1)L + (\phi_1 - \phi_2)L^2 + \phi_2 L^3]\omega_t. \end{aligned} \tag{13}$$

The matrix of interest, namely $\mathbf{A}$ of (11) is then given by

$$\mathbf{A} = \begin{bmatrix} 2(2B-3D+\phi_2) & 2 & 2(B+D-\phi_2) & 2C & 2\phi_1(1-\phi_2) & 2 \\ 3B-6D+2\phi_2 & 0 & -B-2D+3\phi_2 & 2\phi_2 & -B & -C \\ 2(B-2D) & 0 & D-3\phi_2 & 0 & 0 & 0 \\ B-2D-\phi_2 & 0 & \phi_2 & -\phi_2 & B+\phi_2 & C \\ -(D+\phi_2) & -1 & 0 & -C & -\phi_1(1-\phi_2) & -1 \\ -\phi_2 & 0 & 0 & -\phi_2 & -\phi_2 & 0 \end{bmatrix} \qquad (14)$$

in which

$$\begin{aligned} B &= 1+\phi_1^2+\phi_2^2 \\ C &= 1+\phi_1+\phi_2 \\ D &= \phi_1+\phi_2-\phi_1\phi_2. \end{aligned}$$

Once again, further details on the derivation of (14) can be found in the Technical Appendix.

Straightforward row operations applied to (14) show that

$$\begin{bmatrix} \gamma_0+2\gamma_4 \\ \gamma_1+\gamma_3+\gamma_5 \\ \gamma_2 \\ \gamma_3-\gamma_5 \\ \gamma_4 \\ \gamma_5 \end{bmatrix} = \begin{bmatrix} 4(B-2D) & 0 & 2(B+D-\phi_2) & 0 & 0 & 0 \\ 4(B-2D) & 0 & -(B+2D-4\phi_2) & 0 & 0 & 0 \\ 2(B-2D) & 0 & D-3\phi_2 & 0 & 0 & 0 \\ B-2D & 0 & \phi_2 & 0 & B+2\phi_2 & C \\ -(D+\phi_2) & -1 & 0 & -C & -\phi_1(1-\phi_2) & -1 \\ -\phi_2 & 0 & 0 & -\phi_2 & -\phi_2 & 0 \end{bmatrix} \begin{bmatrix} \sigma_\tau^2 \\ \sigma_c^2 \\ \sigma_s^2 \\ \sigma_{\tau c} \\ \sigma_{\tau s} \\ \sigma_{cs} \end{bmatrix}. \qquad (15)$$

The system of (15) exhibits three characteristics that are important for identification when $\phi_2 \neq 0$. Firstly, the first three equations show that the variance parameters σ_τ^2 and σ_s^2 are over-identified, since there are three pieces of information ($\gamma_0 + 2\gamma_4$, $\gamma_1 + \gamma_3 + \gamma_5$ and γ_2) available for these two parameters. Secondly, since further row operations can be used to reduce any one of these first three rows of $\mathbf{A}$ to contain only zeros, the rank condition for all parameters in σ to be identified is not satisfied; the matrix $\mathbf{A}$ has rank less than 6. In terms of the original parameters, it can be seen that the linear dependence is

$$[2\gamma_2 - \gamma_1 - \gamma_3 - \gamma_5] = \frac{[1 + \phi_1^2 + \phi_2^2 + 4\phi_1 - 6\phi_2 - 4\phi_1\phi_2]}{2[1 + \phi_1^2 + \phi_2^2 + 2\phi_2]}[\gamma_0 + 2\gamma_4 - 2\gamma_2].$$

The third characteristic of (15) is that (when $\phi_2 \neq 0$) its rank is five when any one of the last three columns is deleted. Therefore, *a priori* specification of the value of any one of the innovation correlations $\sigma_{\tau c}$, $\sigma_{\tau s}$ or σ_{cs} is sufficient for the remaining elements of $\mathbf{Q}$ to be identified.

As an aside, the crucial role played by $p > 1$ is evident in (14), since $\phi_2 = 0$ yields an $\mathbf{A}$ in (14) whose final row contains only zeros, implying the rank is at most 5 and the model as a whole is not identified. Indeed, combined with the nature of the first three rows, it can be seen that the rank is 4; the situation is then similar to the case of a white noise cycle, considered in the preceding subsection.

To summarize, some properties of the individual components in the general correlated trend-cycle-seasonal model of (1) to (5) can be obtained from observations on y_t, but a decomposition for quarterly data cannot be achieved without at least one further restriction. To be more specific, with an $AR(2)$ cycle, one covariance restriction is required for estimates to be obtained for the remaining parameters; should the AR cycle order have $p < 2$, then two restrictions are required. Although the specification of such restrictions may appear to be problematic, it should be recalled that the usual uncorrelated innovation model is more restrictive and although the over-identifying restriction(s) of that model can be tested, such a test is rarely conducted in practice.

4 Simulations

A simulation study is undertaken to examine the empirical implications of the identification issues discussed in the previous section. The data generating process (DGP) is given by (1) to (5) with $p = 2$, in which case one covariance restriction is required for identification. We set

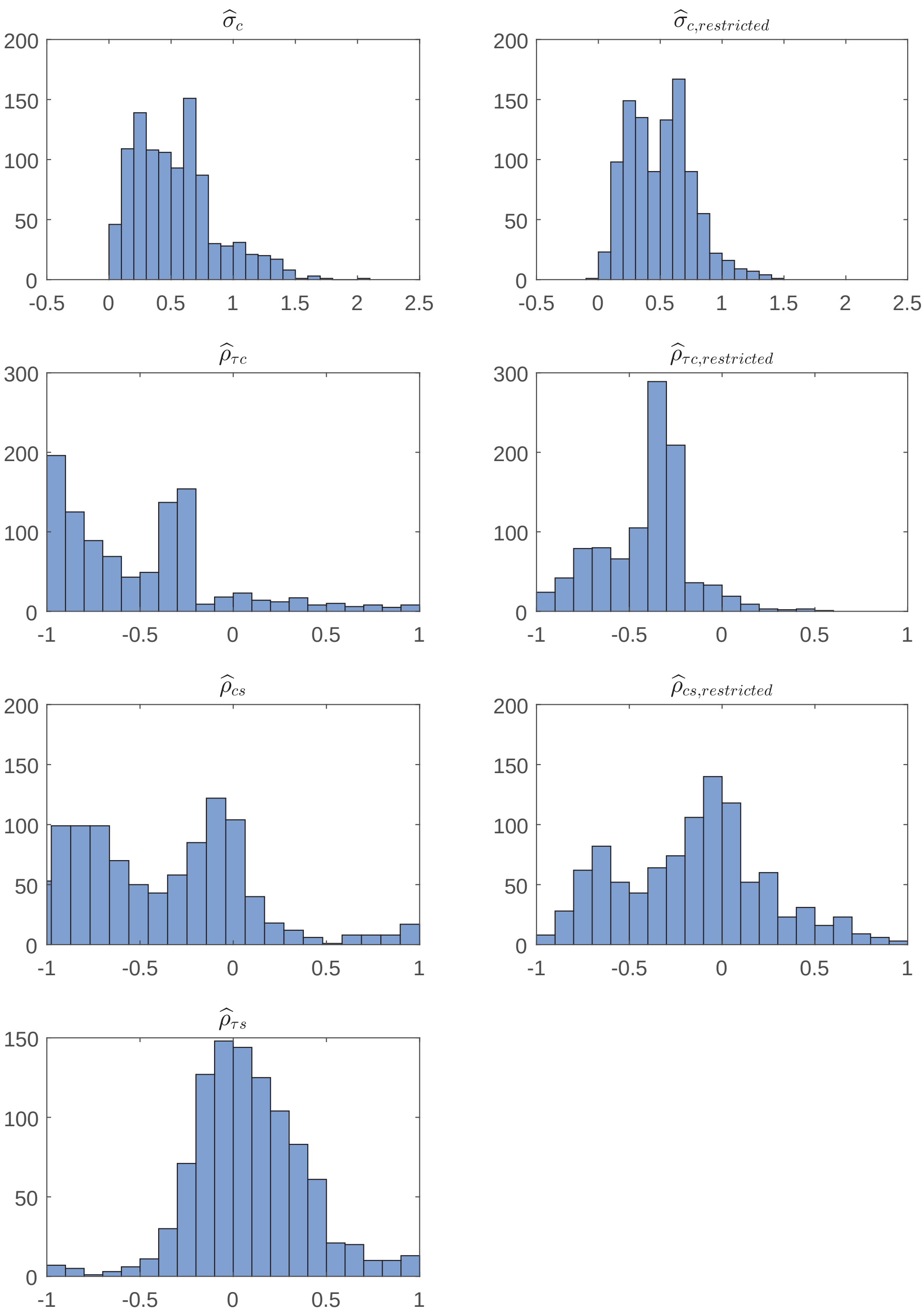
$\phi_1 = 1.35$ and $\phi_2 = -0.5$ in the AR process for the c_t, implying stationary cyclical variation with a periodicity of 21 quarters. For the covariance matrix, we set innovation standard deviations as $\sigma_\mathcal{T} = 1.24$, $\sigma_c = 0.75$, $\sigma_s = 0.1$ and, using an obvious notation for correlations, $\rho_{\mathcal{T}c} = -0.85$, $\rho_{\mathcal{T}s} = 0$ and $\rho_{cs} = -0.3$; hence the covariances are $\sigma_{\mathcal{T}c} = -0.85 \times \sigma_\mathcal{T} \times \sigma_c = -0.7905$, $\sigma_{\mathcal{T}s} = 0$, and $\sigma_{cs} = -0.3 \times \sigma_c \times \sigma_s = -0.0225$. The covariance parameter values for the trend and cycle components (including correlation) are close to those estimated by MNZ for US GDP, while σ_s is chosen to be smaller than for these other components as seasonality is usually observed to evolve relatively slowly over time. A negative cycle-seasonal correlation is implied by the economic arguments and empirical findings of Cecchetti and Kashyap (1996). Finally, the trend-seasonal correlation is set to zero[3], and hence (from the discussion of subsection 3.2) all parameters are (theoretically) identified when this restriction is imposed in estimation.

Maximum likelihood estimation uses GAUSS software[4] with constraints on the AR estimates of $-1 < \widehat{\phi}_1 + \widehat{\phi_2} < 1$ for stationarity and the estimated covariance matrix $\widehat{\mathbf{Q}}$ positive definite. The sample size is 300 observations, corresponding to 75 years of quarterly data, and 1000 replications are performed.

Figure 1 provides results for σ_c, $\rho_{\mathcal{T}c}$, $\rho_{\mathcal{T}s}$ and ρ_{cs} in the form of histograms, both when estimating a general covariance matrix (left-hand column) and imposing $\rho_{\mathcal{T}s} = 0$ (right-hand column). Results are not shown for $\sigma_\mathcal{T}$, σ_s, ϕ_1, and ϕ_2 as the analysis of Section 3 shows that these are identified irrespective of the correlation assumption and it may be noted that the general shapes of the histograms for these parameters are similar across the two cases.

With no restriction, it is seen that the largest mass for $\widehat{\rho}_{\mathcal{T}c}$ is concentrated around -1, implying (spurious) perfect negative correlation between trend and cycle, with $\widehat{\rho}_{cs}$ displaying a similar tendency to bunch at this lower bound. Although Wada (2012) considers a misspecified nonstationary trend-cycle model for a stationary data generating process, he also finds spurious perfect negative estimated correlation for the innovations. Perhaps surprisingly, the histogram for $\widehat{\rho}_{\mathcal{T}s}$ is, at least superficially, relatively well behaved, while that for $\hat{\sigma}_c$ is fairly flat across a range of possible values from 0.1 to 0.8.

Figure 1: Simulation results of the estimated parameters in the UC model



Notes. The panels of the figure show histograms for selected parameters of a UC model, estimated with an unrestricted covariance matrix (left-hand column) and imposing the true restriction $\rho_{\tau s} = 0$ (right-hand column). See the text for other parameter values of the DGP. The sample size is 300 and 1000 replications are performed.

Imposing the true restriction $\rho_{\tau s} = 0$ in estimation, the right-hand panel of Figure 1 no longer shows a large mass of $\widehat{\rho}_{\tau c}$ or $\widehat{\rho}_{cs}$ values close to -1. In particular, these histograms are now more bell-shaped. However, interestingly, $\hat{\sigma}_c$ largely retains its properties from the unidentified case.[5]

The results in this section show that identification requires careful consideration in the correlated trend-cycle-seasonal model. Hidden dependence between the autocovariances renders the correlations unidentified in the plausible model we study, frequently resulting in spurious perfect negative correlations in estimation. Consequently, a perfect estimated correlation needs to be interpreted with care. However, when it is known that one correlation is zero (and hence the model is identified), imposition of this restriction yields estimators with satisfactory properties.

5 Applications

In this section the trend-cycle-seasonal unobserved component model is applied to two important quarterly macroeconomic time series, namely real UK household consumption expenditure and US non-farm employment.[6] In order to make direct comparisons with the results of MNZ and other studies that examine trend-cycle decompositions in a UC framework for the US economy, we would have liked to examine US GDP. Unfortunately, however, that series is not available in a seasonally unadjusted form.[7]

The model applied is again given by (1) to (5) with $p = 2$. As discussed in Section 3, the parameters of the specification with uncorrelated components is over-identified, but at least one restriction is required for identification when a more general covariance structure is permitted. In each case we examine the uncorrelated component model together with other specifications. However, for ease of interpretation, the estimated model is parameterised in terms of correlations ($\rho_{\tau c}$, $\rho_{\tau s}$, ρ_{cs}) and standard deviations rather than the corresponding covariances and variances. Estimation is undertaken by constrained Maximum likelihood in GAUSS using the CMLMT procedure, with any correlation parameters estimated being initialised at zero.

5.1 UK household consumption expenditure

The characteristics of seasonal UK consumption expenditure have provided an important impetus for understanding the long-run properties of economic time series and stimulated some of the early literature on unit roots and cointegration; see, in particular Davidson, Hendry, Srba and Yeo (1978) and Hylleberg, Engle, Granger and Yoo (1990). In line with those studies, we analyse real seasonally unadjusted UK household final consumption expenditure imposing both zero frequency and seasonal unit roots, but adopt the UC framework in order to examine the possibility that the component disturbances may be correlated. The available quarterly data starts in 1955Q1 and our analysis extends from that date to 2016Q4. As usual, the logarithmic transformation is applied prior to further analysis, with the log values also multiplied by 100 to facilitate interpretation of fluctuations in terms of percentage movements.

Table 1 provides results for a range of estimated models[8], while Figure 2 provides the data (top graph in each column) and estimated components for selected cases. Consider first the conventional uncorrelated UC model. This yields a relatively smooth estimated trend, which is seen in Figure 2 and also shown by in the relatively small value of $\widehat{\sigma}_\tau$ for this model in Table 1. However, the estimated cyclical component exhibits relatively large fluctuations over the latter part of the series, being more than 8% above trend in 2005 and declining to nearly 10% below trend at the end of the sample. On the other hand, seasonal fluctuations decline in magnitude over time. Since seasonality evolves only slowly over time, largely the same quarterly pattern repeats each year, with consumption being highest in the Christmas quarter and lowest in the first quarter.

As discussed in Section 3, if the cycle component is white noise or $AR(1)$, then the uncorrelated UC model has a single overidentifying restriction, whereas with an $AR(2)$ cycle the model imposes two more restrictions than required for (exact) identification. In the former case, separation of $\sigma_{\tau c}$ and σ_c^2 requires the value of $\rho_{\tau c}$ to be specified *a priori*, in addition to $\rho_{\tau s}$ or ρ_{cs}. Although the estimated $AR(2)$ coefficient, $\widehat{\phi}_2$ is not significant (at the usual levels) for the uncorrelated UC model in Table 1, it becomes highly significant when only one of the trend correlations ($\rho_{\tau c}$ or $\rho_{\tau s}$) is specified as zero.

Table 1: Estimation Results for UK Household Consumption

		Restriction(s) Imposed			
Parameter	All $\rho_{ij} = 0$	$\rho_{\tau c} = 0$	$\rho_{\tau s} = 0$	$\rho_{cs} = 0$	$\rho_{cs} = -0.99$
σ_τ	0.0936 (0.4370)	0.5959 (0.2634)	0.7904 (0.1545)	1.5581 (0.3801)	1.0895 (0.2153)
σ_c	1.0634 (0.1167)	0.5112 (0.1880)	0.3221 (0.0994)	0.7690 (0.7306)	1.2524 (0.1471)
σ_s	0.4808 (0.0573)	0.5278 (0.0614)	0.5361 (0.0600)	0.5096 (0.0602)	0.5022 (0.0492)
$\rho_{\tau c}$	0 (NA)	0 (NA)	−0.0260 (0.0762)	−0.8035 (0.1558)	−1.0000 (0.0002)
$\rho_{\tau s}$	0 (NA)	0.4174 (0.3978)	0 (NA)	−0.1676 (0.1311)	0.9901 (0.0213)
ρ_{cs}	0 (NA)	−0.9087 (0.1827)	−0.9996 (0.0020)	0 (NA)	−0.99 (NA)
μ	0.6867 (0.0275)	0.6875 (0.0456)	0.6833 (0.0541)	0.6739 (0.0992)	0.7439 (0.0142)
ϕ_1	1.0877 (0.1369)	1.6022 (0.1609)	1.7611 (0.1019)	1.1167 (0.4086)	1.4740 (0.0111)
ϕ_2	−0.1026 (0.1379)	−0.6103 (0.1605)	−0.7684 (0.1017)	−0.2666 (0.4069)	−0.4850 (0.0113)
Log Lik.	−473.110	−467.248	−467.620	−471.541	−459.917
$2(LL - LL_0)$		11.724	10.998	3.138	15.528
p-value		0.0028	0.0041	0.2083	0.0004

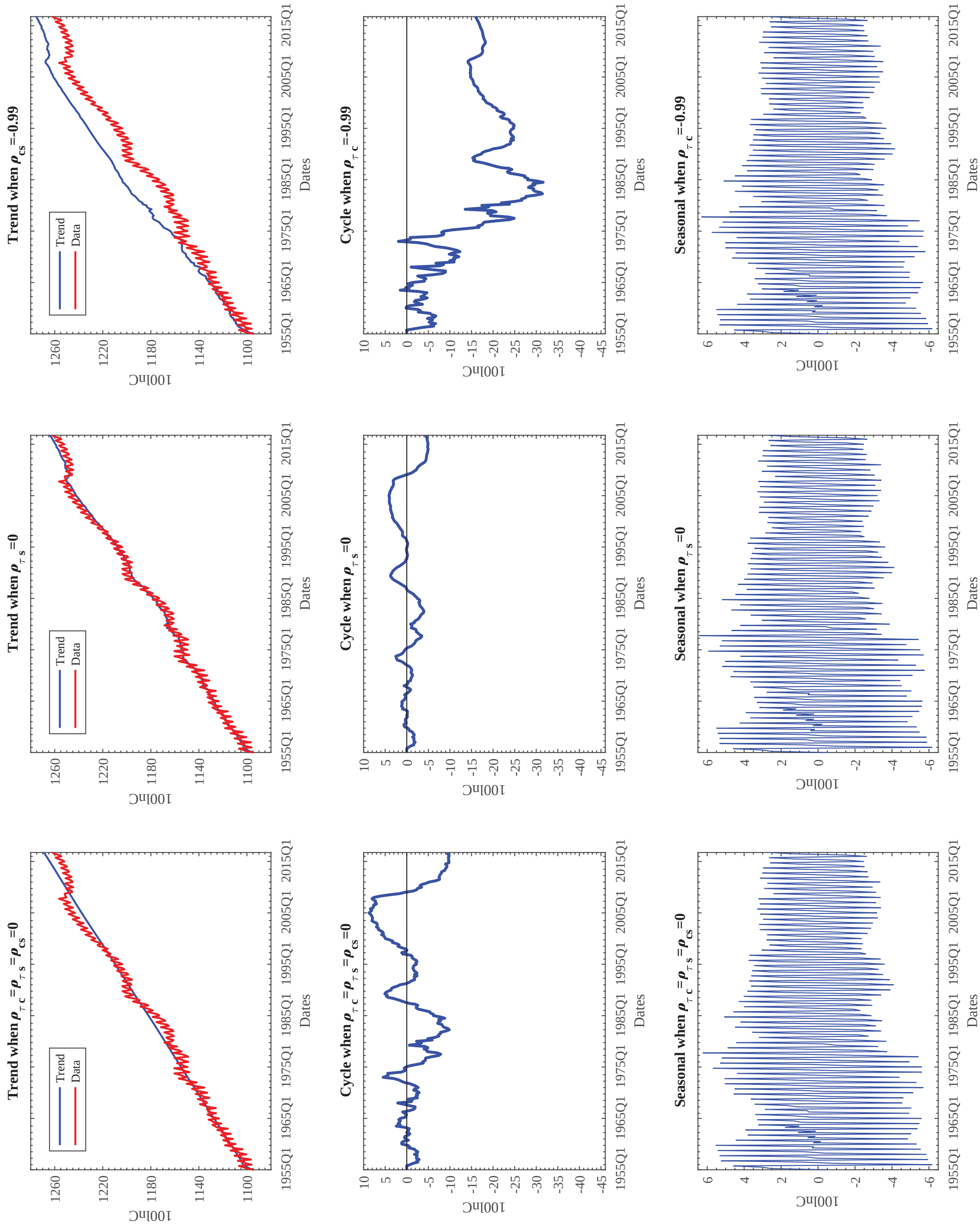
Notes: Values in parentheses are standard errors; NA indicated not applicable, as the parameter value is specified *a priori*; $2(LL - LL_0)$ gives twice the difference between value of the log likelihood and that of the corresponding restricted model (the uncorrelated UC model for all except the final model estimated) denoted LL_0; for the final model the corresponding restricted model the correlation ρ_{cs} is restricted to -0.99, $\rho_{\tau c}$ and $\rho_{\tau s}$ to zero; p-value is computed by comparing $2(LL - LL_0)$ to a χ^2_2 distribution.

Also, both models that impose a single trend correlation restriction yield increases in the log likelihood that are significant at 0.5% (according to an asymptotic χ^2 distribution with 2 degrees of freedom) compared with the uncorrelated UC baseline model. Indeed, these two models are similar in practice, since neither $\rho_{\tau c}$ nor $\rho_{\tau s}$ is significant when one is specified as nonzero and the other estimated. Hence these models yield effectively the same log likelihood value and imply that the correlation between the cycle and seasonal disturbances is very strong and negative. Due to their similarity (including estimated component series) only the case with $\rho_{\tau s} = 0$ is included in Figure 2 (second column). Also note that the model specified with $\rho_{cs} = 0$ as the single restriction in Table 1 is statistically dominated by others, since its log likelihood improves only marginally on the uncorrelated UC model.

Compared with the uncorrelated UC model, the model with $\rho_{\tau s} = 0$ has a more volatile trend (compare the estimates of σ_τ in Table 1 and the extent to which the trend series track the data in Figure 2), while the cycle is very substantially less volatile. Overall, the implied dates of so-called growth cycle recessions (that is, periods with negative estimated cycle values in relation to the trend) do not generally change markedly in comparison with the uncorrelated UC case, although the cycles are typically more marked for the uncorrelated UC model. Nevertheless, the 1990s recession is barely discernible for the correlated component model, but cycle values more than 2% below trend are estimated for the uncorrelated UC model.

In the light of the $\widehat{\rho}_{cs}$ values obtained from other models, the final model of Table 1 specifies $\rho_{cs} = -0.99$, rather than imposing any zero restriction. In statistical terms, the results are impressive, with the log likelihood showing an increase that is significant at 0.001% compared with the corresponding restricted model (namely with $\rho_{\tau c} = \rho_{\tau s} = 0$ and $\rho_{cs} = -0.99$). Further, the estimates imply that a version of the single source of error (SSE) model, in which all component disturbance correlations are ± 1, is supported by the data. Despite this statistical support, Figure 2 shows that the estimated trend and cycle components are not plausible in economic terms, with consumption below trend and the cycle taking large negative values over much of the period since the 1960s. This may imply that the individual

Figure 2: Estimated trend, cycle and seasonal components in UK consumption



Notes. The first column shows estimated trend, cycle and seasonal components in U.K. consumption for the uncorrelated UC model, in the second column we impose $\rho_{\tau s} = 0$, and in the third $\rho_{cs} = -0.99$. The estimated seasonal components in the bottom row vary by the quarter, which at times results in different intensity colors.

trend, cycle and seasonal components are so inter-linked for this series that a decomposition is economically meaningless for this series. Such a view is compatible with the conclusion of Osborn, Chui, Smith and Birchenhall (1988) that UK consumption is periodically integrated, implying an inherent connection between long-run unit root and intra-year seasonal dynamics.

Despite the different estimated disturbance correlations seen in Table 1, it is notable that both $\widehat{\sigma}_s$ and the extracted seasonal component time series change relatively little across all models examined. In that sense, seasonality is robust to the UC specification and seasonal adjustment might be considered appropriate. However, the model in Table 1 where seasonality is largely uncorrelated with the other components (as $\rho_{cs} = 0$ is imposed and $\widehat{\rho}_{\tau s}$ is small) is statistically dominated by other specifications. From a slightly different perspective, the presence of correlations across the components will imply that seasonality contains information relevant for trend and cycle estimation.

5.2 US non-farm payroll employment

US employment data are available seasonally unadjusted from 1948 and we analyse quarterly data over 1948Q1 to 2016Q1. Results are reported in Table 2 for models embodying differing correlation assumptions, with the conventional uncorrelated UC model again providing a baseline. Since the $AR(2)$ coefficient is significant, the uncorrelated UC specification imposes two overidentifying restrictions. Only a single correlation restriction is required for identification and we choose $\rho_{\tau s} = 0$ in view of previous literature which provides evidence of trend-cycle and cycle-seasonal correlations for output and related series (discussed above). In common with UK consumption examined in the previous subsection, the additional restrictions imposed by the conventional model are strongly rejected by an asymptotic log likelihood test.

It is interesting that, as for UK consumption in the preceding subsection, the imposition of $\rho_{\tau s} = 0$ leads to an estimated correlation lying at the -1 boundary and the other being numerically small and statistically insignificant. However, for employment it is the trend-cycle correlation which is estimated at the -1 boundary, rather than the cycle-seasonal correlation.

Table 2: Quarterly US Non-farm Payroll Employment: Estimation Results

Parameter	All $\rho_{ij} = 0$	Restriction(s) Imposed $\rho_{\tau s} = 0$	$\rho_{\tau c} = -0.99$
σ_τ	0.0156 (0.0297)	1.1896 (0.6015)	0.7531 (0.1399)
σ_c	0.5440 (0.0396)	1.5076 (0.7538)	0.9751 (0.1863)
σ_s	0.1557 (0.0169)	0.1465 (0.0260)	0.1113 (0.0141)
$\rho_{\tau c}$	0 (NA)	-1.0000 (0.0001)	-0.99 (NA)
$\rho_{\tau s}$	0 (NA)	0 (NA)	0.9995 (0.0180)
ρ_{sc}	0 (NA)	-0.0065 (0.0168)	-0.9914 (0.1755)
μ	0.4611 (0.0264)	0.4817 (0.0078)	0.4830 (0.0100)
ϕ_1	1.6292 (0.0596)	1.3823 (0.1999)	1.5351 (0.0916)
ϕ_2	-0.6360 (0.0600)	-0.3926 (0.2022)	-0.5449 (0.0924)
Log Lik.	-321.420	-313.582	-301.796
$2(LL - LL_0)$		15.676	26.588
p-value		0.0004	< 0.00001

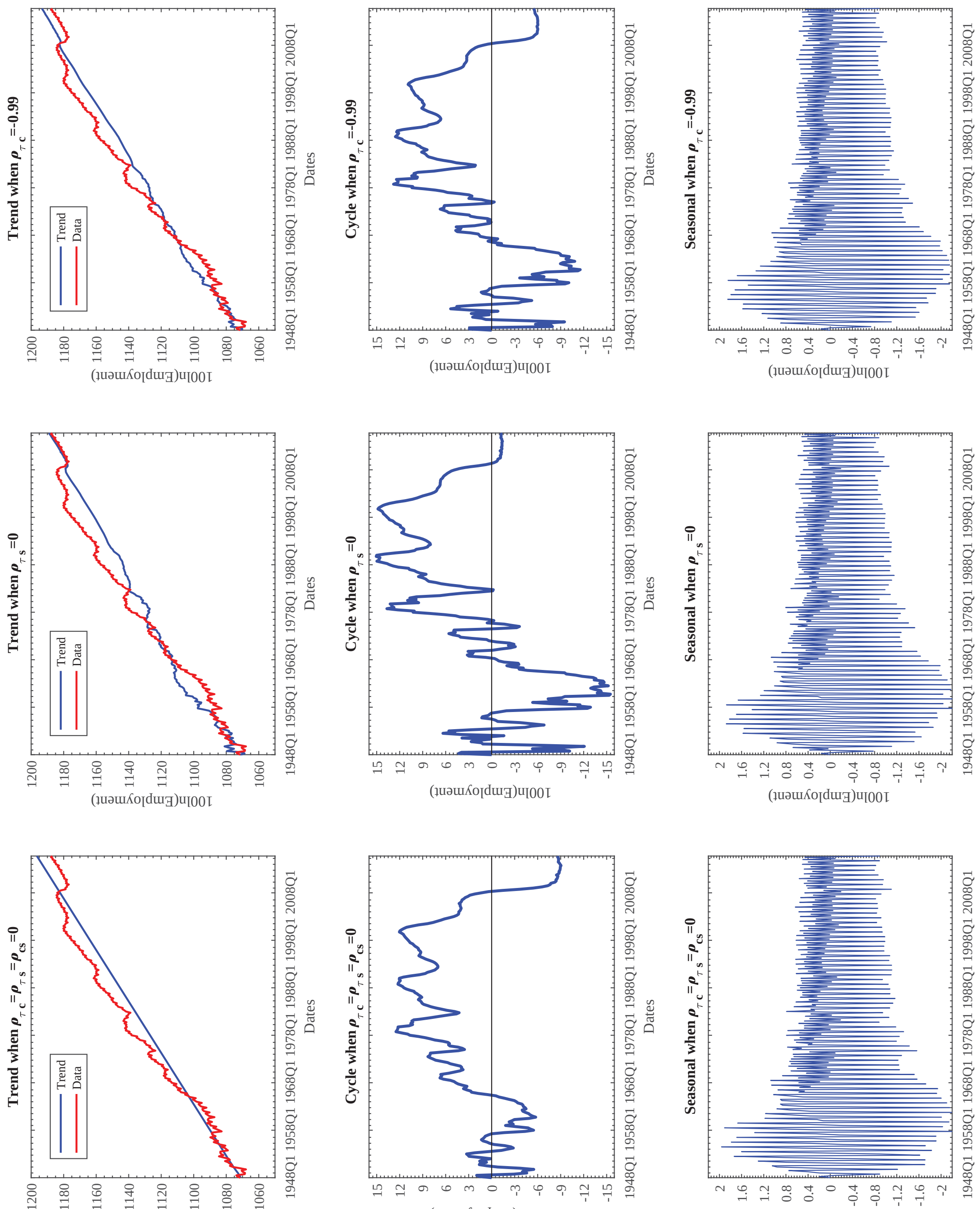
Notes: Values in parentheses are standard errors; NA indicated not applicable, as the parameter value is specified *a priori*; $2(LL - LL_0)$ gives twice the difference between value of the log likelihood and that of the corresponding restricted model (the uncorrelated UC model for all except the final model estimated) denoted LL_0; for the final model the corresponding restricted model the correlation $\rho_{\tau c}$ is restricted to -0.99, $\rho_{\tau s}$ and ρ_{cs} to zero; the p-value is computed by comparing $2(LL - LL_0)$ to a χ^2_2 distribution.

This difference could be associated with the strength and nature of the seasonality in the two series, which is relatively less marked for the employment series (see Figure 3). The final model in Table 2 then imposes a trend-cycle innovation correlation of -0.99, with the results again pointing to an SSE specification being preferred from the statistical perspective over the other specifications. Also as for UK consumption in Table 1, the estimate of σ_s is fairly robust across estimated models, but those for σ_τ and σ_c (especially the former) are not.

Figure 3 displays the estimated components for the three models of Table 2. It is notable that the uncorrelated UC model implies that employment is predominately above trend over an extended period until the Great Recession, with the level subsequently below trend. However, imposing $\rho_{\tau s} = 0$ indicates that the estimated trend largely coincides with observed levels since 2010. The model based on $\rho_{\tau c} = -0.99$ is intermediate between these two cases, with the recent employment gap being smaller than implied by the uncorrelated UC model. In other words, the restrictions imposed on the disturbance correlations in the UC model has substantive implications for trend estimates and consequently for estimates of the employment gap, echoing the findings of MNZ, Morley and Piger (2012), and others.

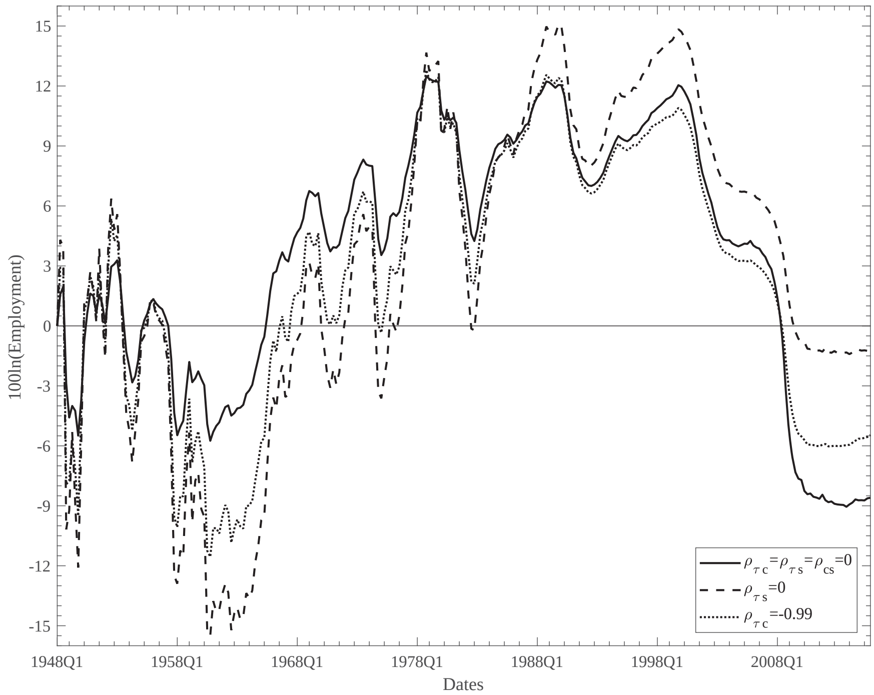
This is seen more clearly in Figure 4, which shows the time series of estimated cycles for the models of Table 2. In general, the timing of employment gap recessions (that is, negative estimated cycle values) differ relatively little across the three specifications, although it is notable that the model with the single restriction $\rho_{\tau s} = 0$ is the only one which detects a recession in the mid-1970s and this specification also differs from the others in dating the Great Recession to start in 2009Q4, one year later than the other specifications. Assumptions made about the disturbance correlations, however, have more striking implications for the amplitude of cyclical movements. In particular, the uncorrelated UC model estimates employment to have been stuck at 8% below trend over an extended period from around 2010, whereas the assumption that trend and seasonal disturbances are uncorrelated (but with $\widehat{\rho}_{\tau c} = -1$) puts the gap at little more than 1% and the SSE model finds this to be 5-6 percent. The extent of these differences imply that employment gaps extracted from UC models should be used with great care in policy making.

Figure 3: Estimated trend, cycle and seasonal components in U.S. employment



Notes. The first column shows estimated trend, cycle and seasonal components in U.S. employment for the uncorrelated UC model, in the second column we impose $\rho_{\tau s} = 0$, and in the third $\rho_{\tau c} = -0.99$. The estimated seasonal components in the bottom row vary by the quarter, which at times results in different intensity colors.

Figure 4: Estimated cycles in U.S. employment



Notes: Solid line: estimated cycle from the zero correlations model; dashed line: estimated cycle from the model with $\rho_{\tau s} = 0$: dotted line: estimated cycle from the model with $\rho_{\tau c} = -0.99$.

It should be noted that these nontrivially different implications are not only a consequence of the trend-cycle correlation (examined by MNZ and others), but also depend on the assumption made about whether seasonality is uncorrelated with the other components. Hence even though the estimated seasonal components for US employment are very similar across specifications (and hence all models would result in very similar seasonally adjusted values), correlations of the seasonal component with the trend and cycle components can substantially alter the apparent characteristics of these other components. For example, policy prescriptions adopted for the US economy could be very different for employment believed to be 8% below trend compared with 1%.

Finally, the model that is central in the paper consists of a random walk with drift specification for the trend, a stationary $AR(2)$ process for the cycle and seasonality in dummy variable form. Although the specification has been used frequently in empirical UC models of e.g. US output, extensions covering the great recession should consider smoother definitions of the trend components, like an I(2) process, the Hodrick-Prescott filter or the alternative recently suggested by Hamilton (2017), which affects all components. We hope to explore this line of research in future work.

6 Conclusion

This paper argues that seasonality is an inherent feature of the dynamic evolution of macroeconomic time series and, as such, should be considered by economists alongside trend and cycle characteristics As discussed by Wright (2013), the sharp downturn associated with the Great Recession has highlighted the importance of the treatment of seasonality and its mistreatment can have important economic implications for analysis of the trend-cycle components.

We therefore extend the unobserved components specification widely used by macroeconomists for quarterly data to also take account of stochastic seasonality. Since distinct streams of previous literature argue on economic and statistical grounds that, on the one hand, innovations to trend and cycle components may be correlated and, on the other, that seasonal and cycle components are related, our general model permits possible nonzero correlations across the innovations for all three components. However, our analysis shows that identification is not a straightforward extension of the trend-cycle case, due to the presence of linear dependencies between the autocovariances in the companion reduced-form ARIMA model. Simulations show estimation of the resulting under-identified model often leads to spurious perfect negative innovation correlations, but imposing the true zero correlation of the data generating process improves estimation.

Although the general correlated unobserved components model is under-identified, nevertheless the conventional uncorrelated UC model is over-identified. Therefore, the commonly-made assumption of uncorrelated innovations is testable. As a minimum, the sensitivity of extracted trend and cycle components to the correlation assumption can be established.

In our applications we examine the role of the correlation assumption for UK quarterly household consumption since 1955 and US quarterly non-farm payroll employment since 1948, finding that the correlation assumption is, indeed, strongly rejected by the data. Imposition of a zero correlation assumption between trend and seasonal innovations leads to an estimated cycle-seasonal correlation of -1 for UK household consumption and an estimated trend-cycle correlation of -1 for the US employment series. The latter outcome is largely in line with (albeit a little stronger than) that found by researchers considering correlated trend-cycle models for seasonally adjusted output data. Interestingly, imposition of the restrictions then effectively yields a single source of error model for both series, in which all three components are driven by a single shock. Put differently, with a perfect negative correlation between cycle and seasonal for UK household consumption or trend and cycle innovations for US employment, the seasonal innovations are also found to be perfectly correlated with the trend and cycle innovations in quarterly employment. Although such perfect correlation may be partly a consequence of estimates 'piling up' at boundary values, the improvements in fit over the uncorrelated UC model are very substantial.

An important aspects of our analysis of employment concerns the sensitivity of the trend and cycle estimates to the effective assumption made about seasonality. Although the estimates of the (filtered) seasonal components are very similar across the three models examined, the trend and cycle estimates are somewhat different in the period following the Great Recession. In particular, the uncorrelated UC model implies a much deeper recession (the cycle values being -8 percent or more from mid-2010) compared with the model whose perfectly correlated trend-cycle innovations are uncorrelated with seasonal innovations (cycle values around -1 percent). The (effective) single source of error model implies that the seasonal component has information about the trend-cycle components, with a post-recession trend intermediate between these other models and a recession with of depth 5 to 6 percent.

One underlying message of our analysis is that if seasonality is correlated with other components of economic time series, then component extraction is statistically difficult. Nevertheless, imposing the conventional uncorrelated component assumption will not only be invalid when such correlation is present, but ignoring seasonality through the use of seasonally adjusted data will throw away important information about the trend and cycle characteristics of primary interest to macroeconomists. An alternative might be to use the seasonal adjustment method without revisions of Abeln and Jacobs (2016).

Technical appendix

A Reduced Form Specification

As explained in the main text, the model examined for quarterly time series data consists of a trend τ_t, a cycle c_t and a seasonal s_t component, with

$$y_t = \tau_t + c_t + s_t \tag{A.1}$$

and

$$\begin{aligned} \tau_t &= \tau_{t-1} + \beta + \eta_t && \text{(A.2)} \\ \phi(L)c_t &= \varepsilon_t && \text{(A.3)} \\ S(L)s_t &= \omega_t && \text{(A.4)} \end{aligned}$$

where the p^{th} order autoregressive (AR) polynomial $\phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$ (L being the usual lag operator) has all roots strictly outside the unit circle and $S(L) = 1 + L + L^2 + L^3$ is the annual summation operator for quarterly data. In practice, we consider $p = 0, 1$ or 2.

The paper analyses the implications for identification of relaxing the usual assumption that the innovations in (A.2) to (A.4) are uncorrelated. Therefore, the paper considers a general positive semi-definite covariance matrix for the innovation vector $\mathbf{v}_t = (\eta_t,\, \varepsilon_t,\, \omega_t)'$, namely where

$$\mathbf{Q} \equiv E[\mathbf{v}_t \mathbf{v}_t'] = \begin{bmatrix} \sigma_\tau^2 & \sigma_{\tau c} & \sigma_{\tau s} \\ \sigma_{\tau c} & \sigma_c^2 & \sigma_{cs} \\ \sigma_{\tau s} & \sigma_{cs} & \sigma_s^2 \end{bmatrix}. \tag{A.5}$$

The assumption for the trend in (A.2) is that this process has a single zero frequency unit root, while $S(L)$ implies that the seasonal component (4) has unit roots at the annual and semi-annual frequencies. Using the usual notation for differences together with the identity $\Delta_4 = (1-L)(1+L+L^2+L^3) = \Delta S(L)$, the process for y_t in (A.1) is seen to require annual differencing ($\Delta_4 = 1-L^4$) to render it stationary. Applying that transformation throughout (A.1) leads to

$$\Delta_4 y_t = S(L)\beta + S(L)\eta_t + \Delta_4 \phi^{-1}(L)\varepsilon_t + \Delta\omega_t$$

and hence

$$\phi(L)\Delta_4 y_t = \phi(L)S(L)\beta + \phi(L)S(L)\eta_t + \Delta_4\,\varepsilon_t + \phi(L)\Delta\omega_t. \tag{A.6}$$

To obtain the reduced form $ARIMA$ specification implied by (A.6), the left-hand side is clearly an $AR(p)$ in $\Delta_4 y_t$, while the right-hand side has constant $\delta = \phi(L)S(L)\beta$ and a moving average (MA) disturbance that arises from the sum

$$\begin{aligned} z_t &= \phi(L)S(L)\eta_t + \Delta_4\,\varepsilon_t + \phi(L)\Delta\omega_t \\ &= (1-\phi_1 L - ... - \phi_p L^p)(1+L+L^2+L^3)\eta_t + (1-L^4)\varepsilon_t \\ &\qquad +(1-\phi_1 L - ... - \phi_p L^p)(1-L)\omega_t. \end{aligned} \tag{A.7}$$

Note that the maximum lags on the trend, cycle and seasonal disturbances in (A.7) are $p+3$, 4 and $p+1$, respectively. Therefore, the maximum lag for which z_t can have a non-zero autocovariance is $\max(p+3,\,4)$ which implies that z_t has a representation as an MA process. This is discussed by Lütkepohl (1984) in the context of aggregating the components of a vector MA process, and hence $z_t = \theta(L)u_t$ is $MA(q)$ where

$$q \leq \max(p+3, 4) \tag{A.8}$$

and u_t is a white noise process. The variance of u_t and the individual MA coefficients in $\theta(L)$ depend in a non-trivial way on the properties of the individual component processes; see Hamilton (1994, pp.102-107) for examples in the context of two uncorrelated MA processes.

To summarise, the reduced form representation of the UC model consisting of (1) to (A.4) in which the covariance matrix of the component disturbances has the general form of (A.5) is

$$\phi(L)\Delta_4 y_t = \delta + \theta(L)u_t, \tag{A.9}$$

which is equation (9) of the main text. Hence $\Delta_4 y_t$ is $ARMA(p, q)$, with AR polynomial $\phi(L)$ from the cycle component and MA order q satisfying (A.8).

B Identification

In the text we write the autocovariances of z_t of (A.7) as

$$\gamma = \mathbf{A}\sigma \tag{B.1}$$

where $\gamma = [\gamma_0, \ldots, \gamma_q]'$, $\sigma = [\sigma_\tau^2, \sigma_c^2, \sigma_s^2, \sigma_{\tau c}, \sigma_{\tau s}, \sigma_{cs}]'$ and $\mathbf{A}$ is a $(q + 1) \times (q + 1)$ matrix.

B.1 White noise cycle

For $p = 0$, (A.7) and (A.8) become

$$\begin{aligned} z_t &= S(L)\eta_t + \Delta_4 \varepsilon_t + \Delta\omega_t \\ &= \eta_t + ... + \eta_{t-3} + \varepsilon_t - \varepsilon_{t-4} + \omega_t - \omega_{t-1} \end{aligned}$$

and

$$q \leq 4.$$

The non-zero autocovariances of z_t are then given by

$$\begin{aligned}
\gamma_0 &= 4\sigma_\tau^2 + 2\sigma_c^2 + 2\sigma_s^2 + 2\sigma_{\tau c} + 2\sigma_{cs} \\
\gamma_1 &= 3\sigma_\tau^2 - \sigma_s^2 - \sigma_{\tau s} - \sigma_{cs} \\
\gamma_2 &= 2\sigma_\tau^2 \\
\gamma_3 &= \sigma_\tau^2 + \sigma_{\tau s} + \sigma_{cs} \\
\gamma_4 &= -\sigma_c^2 - \sigma_{\tau c} - \sigma_{cs}.
\end{aligned} \tag{B.2}$$

Note that $q = 4$ except for the special case $\sigma_c^2 = -(\sigma_{\tau c} + \sigma_{cs})$. Expression (12) of the main text provides $\mathbf{A}$ of (11) for the matrix representation of the system (B.2).

B.2 AR(2) cycle

For $p = 2$, (A.7) and (A.8) become

$$\begin{aligned}
z_t &= [1 + (1-\phi_1)L + (1-\phi_1-\phi_2)L^2 + (1-\phi_1-\phi_2)L^3 - (\phi_1+\phi_2)L^4 - \phi_2 L^5]\eta_t \\
&\quad + [1 - L^4]\varepsilon_t + [1 - (1+\phi_1)L + (\phi_1-\phi_2)L^2 + \phi_2 L^3]\omega_t.
\end{aligned} \tag{B.3}$$

and

$$q \leq 5.$$

It is straightforward but somewhat tedious to show for this case that z_t has autocovariances

$$\begin{aligned}
\gamma_0 &= 2[2+2\phi_1^2+2\phi_2^2-3\phi_1-2\phi_2+3\phi_1\phi_2]\sigma_\tau^2+2\sigma_c^2+2[1+\phi_1^2+\phi_2^2+\phi_1 \\
&\quad -\phi_1\phi_2]\sigma_s^2+2[1+\phi_1+\phi_2]\sigma_{\tau c}+2\phi_1(1-\phi_2)\sigma_{\tau s}+2\sigma_{cs} \\
\gamma_1 &= [3+3\phi_1^2+3\phi_2^2-6\phi_1-4\phi_2+6\phi_1\phi_2]\sigma_\tau^2-[1+\phi_1^2+\phi_2^2+2\phi_1-\phi_2-2\phi_1\phi_2]\sigma_s^2 \\
&\quad +2\phi_2\sigma_{\tau c}-[1+\phi_1^2+\phi_2^2]\sigma_{\tau s}-[1+\phi_1+\phi_2]\sigma_{cs} \\
\gamma_2 &= 2[1+\phi_1^2+\phi_2^2-2\phi_1-2\phi_2+2\phi_1\phi_2]\sigma_\tau^2+[\phi_1-2\phi_2-\phi_1\phi_2]\sigma_s^2 \qquad \text{(B.4)} \\
\gamma_3 &= [1+\phi_1^2+\phi_2^2-2\phi_1-3\phi_2+2\phi_1\phi_2]\sigma_\tau^2+\phi_2\sigma_s^2-\phi_2\sigma_{\tau c} \\
&\quad +[1+\phi_1^2+\phi_2^2+\phi_2]\sigma_{\tau s}+[1+\phi_1+\phi_2]\sigma_{cs} \\
\gamma_4 &= -[\phi_1+2\phi_2-\phi_1\phi_2]\sigma_\tau^2-\sigma_c^2-[1+\phi_1+\phi_2]\sigma_{\tau c}-\phi_1[1-\phi_2]\sigma_{\tau s}-\sigma_{cs} \\
\gamma_5 &= -\phi_2\sigma_\tau^2-\phi_2\sigma_{\tau c}-\phi_2\sigma_{\tau s}.
\end{aligned}$$

Analogously to $p=0$ above, $q \leq \max(p+3,4)$ takes its maximum value (now 5) except for the special case $\sigma_c^2 = -(\sigma_{\tau c}+\sigma_{cs})$. Thus, in general, z_t is $MA(5)$.

To simplify the expressions in (B.4) a little, in the text we define

$$\begin{aligned}
B &= 1+\phi_1^2+\phi_2^2 \\
C &= 1+\phi_1+\phi_2 \\
D &= \phi_1+\phi_2-\phi_1\phi_2.
\end{aligned}$$

Hence the system of autocovariances can be written as

$$\begin{aligned}
\gamma_0 &= 2[2B - 3D + \phi_2]\sigma_\tau^2 + 2\sigma_c^2 + 2[B + D - \phi_2]\sigma_s^2 + 2C\sigma_{\tau c} + 2\phi_1(1 - \phi_2)\sigma_{\tau s} + 2\sigma_{cs} \\
\gamma_1 &= [3B - 6D + 2\phi_2]\sigma_\tau^2 - [B + 2D - 3\phi_2]\sigma_s^2 + 2\phi_2\sigma_{\tau c} - B\sigma_{\tau s} - C\sigma_{cs} \\
\gamma_2 &= 2[B - 2D]\sigma_\tau^2 + [D - 3\phi_2]\sigma_s^2 \\
\gamma_3 &= [B - 2D - \phi_2]\sigma_\tau^2 + \phi_2\sigma_s^2 - \phi_2\sigma_{\tau c} + [B + \phi_2]\sigma_{\tau s} + C\sigma_{cs} \\
\gamma_4 &= -[D + \phi_2]\sigma_\tau^2 - \sigma_c^2 - C\sigma_{\tau c} - \phi_1[1 - \phi_2]\sigma_{\tau s} - \sigma_{cs} \\
\gamma_5 &= -\phi_2\sigma_\tau^2 - \phi_2\sigma_{\tau c} - \phi_2\sigma_{\tau s}.
\end{aligned} \tag{B.5}$$

Expression (14) of the main text provides the matrix $\mathbf{A}$ of (11) for this system of equations.

Notes

[1]More fundamentally, Anderson and Moore (1979, pp.230-234) show that any UC model has a SSE representation. However, the components of such an implied SSE representation may not have forms that are plausible to economists. In contrast, we begin from widely used component specifications.

[2]Although not explicitly drawn out, McElroy and Maravall (2014) effectively also come to this conclusion for the same model as we examine here.

[3]Note we could also specify a DGP with zero $\rho_{\tau c}$ or ρ_{cs}, but $\rho_{\tau s} = 0$ appears the most plausible in that previous analyses have found evidence of nonzero trend-cycle and cycle-seasonal correlations.

[4]Parameter estimates are retained only if the estimation ends as "normal convergence" and the number of iterations does not exceed 1000.

[5]More detailed simulation analysis than possible here would be required to establish how the distribution of this estimator is affected by the imposition of covariance restrictions for other realistic sets of parameter values.

[6]UK household final consumption expenditure is a chained volume measure, reference year 2013, published by the Office for National Statistics (series ABPB, not seasonally adjusted) in the United Kingdom Economic Accounts time series dataset. US non-farm payroll employment is obtained form the Bureau of Labor Statistics (series ID CEU0000000001 on their webpage) with the monthly series converted to quarterly by taking the final month of each quarter.

[7]To quote Wright (2013, p.79) "amazingly, the Bureau of Economic Analysis stopped releasing NSA GDP data some years ago, as a cost-cutting measure."

[8]Although standard errors are included for all estimated parameters, these may be unreliable when the estimated values lie close to a boundary of the permissable range, including for correlation estimates close to ± 1.

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