Working Papers in Trade and Development

Directed technical change and the British Industrial Revolution

David I. Stern
Crawford School of Public Policy
The Australian National University

John C. V. Pezzey
Fenner School of Environment and Society
The Australian National University

Yingying Lu
School of Economics, Shanghai University, China and
CSIRO Australia,

December 2020

Working Paper No. 2020/29

Arndt-Corden Department of Economics
Crawford School of Public Policy
ANU College of Asia and the Pacific

This Working Paper series provides a vehicle for preliminary circulation of research results in the fields of economic development and international trade. The series is intended to stimulate discussion and critical comment. Staff and visitors in any part of the Australian National University are encouraged to contribute. To facilitate prompt distribution, papers are screened, but not formally refereed.

Copies are available at https://acde.crawford.anu.edu.au/acde-research/working-papers-trade-and- development.

Directed Technical Change and the British Industrial Revolution

David I. Stern,[*] John C. V. Pezzey,[**] and Yingying Lu[***]

13 December 2020

Abstract: We build a directed technical change model where one intermediate goods sector uses a fixed quantity of biomass energy ("wood") and another uses coal at a fixed price, matching stylized facts for the British Industrial Revolution. Unlike previous research, we do not assume the level or growth rate of productivity is inherently higher in the coal-using sector. Analytically, greater initial wood scarcity, initial relative knowledge of coal-using technologies, and/or population growth will boost an industrial revolution, while the converse may prevent one forever. An industrial revolution, with eventual dominance by the coal-using sector, is the model's main dynamic outcome, but not inevitable if inter-good substitutability is high enough. Empirical calibration for 1560-1900 produces historically plausible results for changes in energy-related variables during British industrialization, and through counterfactual simulations confirms that it was the growing relative scarcity of wood caused by population growth that resulted in innovation to develop coal-using machines.

Keywords: Economic growth, economic history, energy, coal, structural change.

JEL Codes: N13, N73, O33, O41, Q43

Acknowledgements: We thank: Prema-Chandra Athukorala, Ed Barbier, Timothy Considine, Roger Fouquet, Daniel Garcia-Macia, Nick Hanley, Wei Jin, Brooks Kaiser, Astrid Kander, Bruno Lanz, Akshay Shanker, Gregor Singer, Olli Tahvonen, Mike Toman, and other participants at seminars at the London School of Economics, the Australian National University, the University of Wyoming, the annual conferences of AARES, AERE, EAERE, and the World Congress of Environmental and Resource Economics.

Funding: We thank the Australian Research Council for support under Discovery grant DP120101088: "Energy Transitions: Past, Present and Future" and the National Natural Science Foundation of China for support under grant number 71703096.

[*] Crawford School of Public Policy, The Australian National University, 132 Lennox Crossing, Acton, ACT 2601, Australia. Corresponding author: E-mail: david.stern@anu.edu.au, Telephone: +61-2-6125-0176, ORCID: 0000-0001-6595-4268

[**] Fenner School of Environment and Society, The Australian National University, 141 Linnaeus Way, Acton, ACT 2601, Australia. ORCID: 0000-0001-6473-9355

[***] School of Economics, Shanghai University, China and CSIRO, Black Mountain, ACT 2601, Australia.

1. Introduction

Did coal play a vital role in the acceleration of British economic growth known as the Industrial Revolution? Economists and historians are divided on the importance of coal in fueling the increase in the rate of economic growth. Many researchers (e.g. Wilkinson, 1973; Wrigley, 1988, 2010; Pomeranz, 2001; Krausmann *et al*., 2008; Allen, 2009, 2012; Barbier, 2011; Kander *et al.,* 2014; Fernihough and O'Rourke, 2014; Gars and Olovsson, 2019) argue that innovations in the use, and growth in the quantity consumed, of coal played a crucial role in driving the Industrial Revolution. By contrast, some economic historians (e.g. Clark and Jacks, 2007; Kunnas and Myllyntaus, 2009) and economists (e.g. Madsen *et al*., 2010) either argue that it was not necessary to expand the use of modern energy carriers such as coal, or do not give coal a central role (e.g. Clark, 2014). This debate matters not just for understanding the history of economic development, but also for assessing the prospects for a global energy transition from fossil fuels to renewables in order to avoid dangerous climate change.

We develop a model that shows both analytically and numerically how the scarcity of biomass energy (referred to here as "wood", which includes both firewood and charcoal) relative to coal in Britain could have directed technical change towards the development of coal-using technologies, resulting in an increase in the rate of economic growth. Our baseline empirical model reproduces several stylized facts of the British Industrial Revolution during 1560-1900, without assuming that the level or growth rate of productivity is inherently higher in the coal-using sector.[1] Our primary contribution is thus to show for the first time how and why the Industrial Revolution took place in a country with increasingly scarce wood and abundant coal, namely Britain, without in some sense "stacking the deck" in coal's favor.

Our second contribution is the discovery of a possible development path we call "Pre-industrial Stagnation", where growth is slow and final output becomes ever more concentrated in the *wood*-using sector. This can happen if the elasticity of substitution between wood-intensive goods and coal-intensive goods is high enough, and if initial conditions are sufficiently in wood's favor.

We use an expanding machine varieties (horizontal innovation) approach to modelling directed technical change, which is appropriate since new types of machines and industrial processes using coal were characteristic of the Industrial Revolution. Though the organization of R&D was undoubtedly different in this period than today, entrepreneurs did carry out deliberate R&D and experiments to develop new products and techniques (Allen, 2009, 241) and, Allen argues, responded strongly to price incentives in their choice of innovations to pursue. Our model has two intermediate goods sectors – the "Malthus" and "Solow" sectors – that use wood and coal inputs, respectively, and sectoral goods are then combined into final output via a constant elasticity of substitution (CES) production function. Each sector also uses labor and sector-specific machine inputs. Unlike almost all previous research discussed below, we do not assume that productivity is inherently higher or faster growing in the Solow than in the Malthus sector. Instead, we assume that wood is supplied perfectly inelastically (i.e. with

[1] We do not attempt to model the period after 1900, as from then on additional energy transitions to oil, electricity etc. come into play.

constant quantity), while coal is supplied perfectly elastically (i.e. at a constant price).[2] In the next section, we show these key asymmetric assumptions about energy supply are broadly consistent with the available historical data.

As in other directed technical change models, when the elasticity of substitution between the two sectoral goods is greater than unity, relative innovation activity is positively related to the relative abundance of the two sector-specific factors. Thus, an increase in the scarcity of wood relative to coal increases the level of innovation in the coal-using Solow sector relative to that in the wood-using Malthus sector. Kander and Stern (2014) show that the elasticity of substitution between biomass and fossil fuel energy was greater than unity in Sweden in the late 19th and early 20th Centuries, and we assume this was the case for wood-intensive and coal-intensive goods in Britain.[3]

In our model, growth is forever unbalanced, and there is an underlying tendency towards the relative growth of the coal-using sector, whereas if the quantities of both coal and wood were fixed there would be a balanced growth path (BGP). But when coal is expandable, long-run behavior of the model depends on the degree of substitutability between the two goods. We will show analytically that if wood is initially sufficiently abundant relative to coal, and substitutability between goods produced using wood or coal is high enough, then the incentives for innovation increasingly favor wood-directed technical change, resulting in technical change being wood-biased forever (Pre-industrial Stagnation). For "medium" substitutability, all development paths undergo an Industrial Revolution – defined as coal-intensive goods output becoming ever more dominant. These paths eventually result in Modern Economic Growth, where coal's price relative to wood is no longer falling but rising while its relative use is also rising, thus exhibiting the upward-sloping relative demand curve that Acemoglu (2007) calls strong (relative) bias. For "high" substitutability, either an Industrial Revolution (and Modern Economic Growth) or Pre-industrial Stagnation can happen, depending on the economy's starting point. Pre-industrial Stagnation generally also exhibits strong bias, where wood's relative price rises together with its relative use.

Using an empirically calibrated simulation of our model, we then show how growing population could have forced up the price of wood, resulting in the shift of innovative activity to the coal-using sector. But one of our counterfactual simulations shows how a combination of abundant initial wood and low population growth would lead to wood-dominant, Pre-industrial Stagnation. So "necessity is the mother of invention" in our model, which is broadly the industrial equivalent of Boserup's (1981) mechanism where technical change in agrarian societies is driven mainly by rising natural resource scarcity.

Of course, our model abstracts from other issues such as Allen's (2009) argument that expensive labor was the reason why coal-directed innovation was profitable in Britain long before it was elsewhere. We also implicitly assume that the British institutional environment was appropriate for accelerating growth to occur, for example by having the well-developed

---

[2] Unlike Hanlon's (2015) study of the effect of the American civil war on British innovation, supply conditions do not change over time in our model, rather the elasticities of supply of the factor inputs are different.

[3] When commenting on relevant literature, we use the various terms that each researcher uses for different energy resources such as fossil fuels, renewable energy, biomass, etc., some of which also differ substantively from the "coal" and "wood" categories used in our model.

patenting system which Madsen *et al.* (2010) find to be econometrically significant.[4] Furthermore, we do not make a distinction between the usefulness of different inventions. Authors such as Mokyr (2009b) and Allen (2009) view "macro-inventions" like the steam engine or coke smelting as having a significant role in the Industrial Revolution (Crafts, 2010). However, it can take more than a century of small improvements ("micro-inventions") for technical efficiency to improve enough for a macro-invention to have a significant macro-economic impact (Allen, 2009; Clark and Jacks, 2007). Modelling technological change as deterministic and incremental, as we do here, rather than stochastic and sometimes revolutionary, therefore arguably misses no vital feature of the Industrial Revolution. Finally, we abstract from other properties of coal relative to wood such as higher energy density per cubic meter, or per hectare of the land used for energy production.

Previous research relevant to our model falls into three areas. First are "unified growth" models, which explain the take-off from Malthusian stagnation (where any technical progress results in population rather than income growth) but do not model fossil fuels explicitly. Hansen and Prescott (2002) have two sectors, with a fixed land input in the agricultural, "Malthus" sector, no natural resource input to the industrial, "Solow" sector, semi-endogenous population growth, and exogenous technical progress that is assumed *a priori* to be much faster in the Solow than in the Malthus sector. As a result, the economy transitions from the Malthus to the Solow sector. Other papers in this vein include O'Rourke *et al*. (2013), who introduce directed technical change in a unified growth model, but with sectors distinguished by high or low labor skills rather than by use of land; and Kögel and Prskawetz (2001), Voigtländer and Voth (2006), and Strulik and Weisdorf (2008), who make assumptions about differences in productivity growth, the capital externality, or the elasticity of consumer demand for the output from agricultural and manufacturing sectors.[5]

The second area of relevant literature comprises papers that do model the effect of fossil fuels on long-run growth (Tahvonen and Salo, 2001; Fröling, 2011; Eren and Garcia-Macia, 2013; Gars and Olovsson, 2019). These researchers all assume, like Hansen and Prescott, that fossil fuel-directed technical change is more rapid or less costly to undertake than renewables-directed technical change. Gars and Olovsson (2019) is particularly relevant, as they allow for an equilibrium with persistent slow growth and biofuel-directed technical change. Their model has a CES function with an elasticity of substitution greater than one combining intermediate energy goods made from fossil or biofuel energy sources and sector-specific machines. This energy aggregate then combines with another intermediate good, made from labor and machines in a CES function with elasticity of substitution of less than one to produce the final output. There is Schumpeterian endogenous technical change, no population growth, and the fossil fuel stock grows exogenously. Crucially, they also assume it is more costly to innovate in the biofuel sector than the fossil fuel sector. For a single-country model, if fossil fuels are available, then the economy uses only fossil fuels and only develops the fossil fuel technology.

---

[4] But see Mokyr (2009a) on the limitations of the patent system.

[5] Lewis (1954) was, of course, the first to develop a two-sector model of the transformation of a pre-industrial economy. He assumed an infinitely elastic supply of labor in the traditional, land-based sector, and that capital was only used in the modern sector. But these assumptions about economies in the first stages of industrialization are not necessarily accurate (Gollin, 2014).

Otherwise, it grows much more slowly by developing the biofuel technology. When there are multiple heterogeneous countries and international trade, if developed countries are much more advanced, they bid up the fossil fuel price and make fossil fuel-using innovation unprofitable in developing economies. Developed countries then follow the fossil fuel growth path and developing countries follow the biofuel-only path. By contrast, in our model, both fuels are always used, and we allow for an endogenous transition from an economy with predominantly biofuel directed technical change to one with fossil fuel directed change and more rapid economic growth.

The third area of relevant literature is empirical work on the historical role of coal in the Industrial Revolution. Clark and Jacks (2007) argue that an industrial revolution could still have happened in a coal-less Britain with only "modest costs to the productivity growth of the economy" (p.68), because the value of coal was only a modest share of British GDP, and they argue that Britain's energy supply could have been greatly expanded, albeit at about twice the cost of coal, by importing wood from the Baltic. Madsen *et al.* (2010) find that coal production in British coalmines had no econometrically significant effect on per-capita output. But both Clark and Jacks (2007) and Madsen *et al*. (2010) do not allow for the dynamic effects of resource scarcity on the rate of innovation. On the other hand, Fernihough and O'Rourke (2014) use geographical analysis to show the importance of access to local coal in driving industrialization and urban population growth, though Kelly *et al*. (2015) provide contradictory evidence on this point. Finally, Kander and Stern (2014) econometrically estimate a model of the transition from biomass energy (mainly wood) to fossil fuel (mainly coal) in Sweden, which shows the importance of this transition in economic growth there. However, they assume exogenous factor-directed technical change.

2. Stylized Facts

Figure 1 shows the evolution of GDP per capita over 20-year periods from 1560-1900.[6] Up to 1660, GDP per capita was flat or declining, after which it grew at an accelerating rate, though the growth rate was quite erratic and in the second half of the 19th Century ranged from 0.8% to 1.9% p.a., which is low by 20th or 21st Century standards.[7]

Figure 2 shows the real prices of coal and charcoal in London and the Western Britain (Allen, 2009). The price of charcoal rose steeply from the beginning of the 17th Century to the late 18th Century after which it appears to level off and fall (Fouquet, 2011). The price of coal though is relatively stable over time in both regions. Clark and Jacks (2007) explain that throughout this period innovation in coal extraction overcame the effects of depletion, resulting

[6] For 1870 to 1900 we use the Composite GDP (E) measure of real GDP at 2006 prices from Hills et al. (2010). From 1540 to 1870 we used the growth rates from Broadberry *et al*.'s (2015) estimate of GDP for Great Britain in constant prices of 1700 to project real GDP back to 1540.

[7] Though an acceleration of the rate of economic growth was a defining feature of the Industrial Revolution, the time path of income (per capita) over the last millennium is still deeply disputed among economic historians (Fouquet and Broadberry, 2015). For example, Clark (2010) estimates English income to have changed very little between pre-industrial times and 1800, but Broadberry *et al*. (2015) estimate that income nearly tripled between 1270 and 1800.

in the long-run supply of coal being highly elastic.[8] Figure 3 shows the energy content of firewood (including charcoal) and coal consumed in England and Wales (Warde, 2007, appendix). Firewood provided about 80% of total fuel in 1560, declining to about 25% by 1700 and to zero by 1850. The absolute quantity of firewood used was fairly constant from about 1560 until 1800. Though timber was increasingly imported to Britain, especially in the 19th Century (Iriarte-Goñi and Ayuda, 2012), there does not seem to have been significant international trade in firewood (Thomas, 1986; Warde, 2007). Coal use increased 700-fold over the period. Though the quantity of firewood used eventually fell to zero during the 19th Century, for simplicity our model will assume that wood use for energy (including charcoal) was constant throughout.

Energy intensity in Britain increased till the end of the 19th Century, after which it declined (Kander *et al*., 2014). From 1720 to 1900 it roughly doubled, but prior to the mid-18th Century it was fairly constant (fig. 4). However, figure 4 also shows that if one includes only coal and wood in the energy aggregate – thus eliminating more than half of the energy used in 1560 – then intensity also rose since the early 17th Century and quadrupled by 1900.

Gentvilaite *et al*. (2015) calculate that Britain's energy cost share declined from around 25% of total costs in 1800 to around 15% in 1900. We do not have sufficient data to estimate the cost share before 1800 consistent with Gentvilaite *et al*.'s (2015) data. However, using "back of the envelope" calculations based on the data shown above, it seems that the cost of energy relative to the GDP may have been constant or rising till the late 17th Century before beginning its decline. Therefore, it is reasonably consistent with history that our model will assume a constant energy cost share.

3. The Model

We assume there are two energy sources – wood and coal – which can both be augmented by technological change. Goods produced using either of the two energy sources are good substitutes for each other. In common with Acemoglu (2002) (hereafter A02), technical change is modeled as an expansion of machine varieties, but as in Acemoglu *et al*. (2012), in addition to intermediate machines and labor, natural resources contribute to production. While only one sector has a resource input in Acemoglu *et al*. (2012), in our model each sector has a resource input – wood or coal. Following our discussion of figures 2 and 3, we assume the wood quantity and coal price are exogenously fixed. The consumer supplies a unit of wood whose supply is regulated by institutions, and coal is available at a fixed marginal cost in terms of final output. Therefore, we do not consider the non-renewable nature of coal – or the renewable nature of wood – explicitly, and neither do we model innovation in extraction. Except for some analytic results and in the Constant Population scenario in section 6, we assume population, and hence the labor force, grows exogenously up to a finite limit, so that the available wood quantity per worker falls. As in Acemoglu *et al*. (2012), we use discrete time and assume that a patent for any variety of machine only lasts one period, here 20 years.[9] We assume that at the beginning

---

[8] We model neither depletion nor innovation in the coal mining sector itself, focusing on innovation in the downstream industrial sectors.

[9] If innovators are granted perpetual patents, then they need to consider the net present value of the stream of future profits when deciding how much to invest in innovation activities. As explained by A02, this decision is

of each period, patents for all existing machine varieties are re-issued at random, meaning that all varieties (new and existing) are produced by monopolistic firms, which maximize only current period profits.[10] The 20-year period also is a convenient time step for the assumption that all machines depreciate fully within one period. As a result, the consumer plays no active role in our model: profit maximization ensures that consumption is maximized and there is no intertemporal investment decision, which greatly simplifies the model. For our dynamic analytical results and all our numerical simulations, we also assume that the asymptotic rate of growth will be constant, to match the Kaldor stylized facts (Kaldor, 1963).

3.1. Production, Prices, and the Allocation of Labor and Energy

Final output, Y, is produced competitively from two intermediate goods, Y_M and Y_S, where the subscripts M and S refer to "Malthus" and "Solow", via a constant elasticity of substitution production function:

$$Y_t = \left[\gamma Y_{M,t}^{\frac{\sigma-1}{\sigma}} + (1-\gamma) Y_{S,t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \quad (1)$$

where $\sigma > 1$ is the elasticity of substitution, $0 < \gamma < 1$ is the distribution parameter, and t indicates the (discrete) time period. Various critical values of σ will be shown later to be important in our analysis. The two goods are produced competitively using the following Cobb-Douglas technologies:

$$Y_{M,t} = \frac{1}{\beta}\left(\int_0^{N_{M,t}} x_{M,t}(j)^{\beta} dj\right) \bar{E}_M^{\alpha} L_{M,t}^{1-\alpha-\beta} \quad (2)$$

$$Y_{S,t} = \frac{1}{\beta}\left(\int_0^{N_{S,t}} x_{S,t}(j)^{\beta} dj\right) E_{S,t}^{\alpha} L_{S,t}^{1-\alpha-\beta} \quad (3)$$

where $0 < \alpha, \beta, \alpha + \beta < 1$.[11] The Malthus sector uses the fixed wood supply, $\bar{E}_M$, and a range $N_{M,t}$ of varieties of wood-using machines as inputs, with each variety j used in amount $x_{M,t}(j)$.

---

then complicated because not only might the interest rate vary over time off a balanced growth path – and in our model a balanced growth path is highly unlikely due to the fixed wood supply – but also the relative prices of the two goods will change over time. This would lead to a complicated dynamic programming problem, which is why A02 focuses on deviations from balanced growth. 20 years is the current length of a UK patent. The 1624 Statute of Monopolies set a 14-year period (Khan and Sokoloff, 2004).

[10] This is similar to the assumption in Acemoglu *et al.* (2012). We could instead assume that when the patent expires each machine variety is produced competitively in all following periods, so that its price equals marginal cost, and newly developed machine varieties will, therefore, be priced higher than older varieties and used in smaller amounts (see Gancia and Zilibotti (2005) and appendix B9 of Acemoglu *et al.* (2012) for similar models). This is what is seen in the real world, where new technologies are expensive and sold in smaller quantities but later become commodified. However, this assumption complicates our analytical model without changing our qualitative results or adding any useful insights.

[11] For analytical tractability, given our model's historically realistic asymmetry between wood and coal supply conditions, we use this Cobb-Douglas form, thus departing from another realistic assumption, namely that the elasticity of substitution between energy and machines is less than 1, as used in previous research (Stern and Kander, 2012; Kander and Stern, 2014). Lemoine (2017) shows how low substitutability between energy and machines avoids an important source of sectoral "lock-in" that may prevent long-run energy transitions; so, generalizing (2)-(3) to the non-Cobb-Douglas case remains a worthwhile topic for further research. However, numerical simulations show that an elasticity less than 1 gives results not much different from those in this paper. Note that for most purposes our β corresponds to $1 - \beta$ in A02.

Asymmetrically, the Solow sector uses an indefinitely expandable coal supply, $E_{S,t}$, and a range $N_{S,t}$ of varieties of coal-using machines as inputs, with each variety used in amount $x_{S,t}(j)$. The initial ranges of machine varieties that can be used with wood and coal, respectively $N_{M,0} > 0$ and $N_{S,0} > 0$, are given as parameters. The numbers of types of machines are also referred to as "knowledge" in the following.

The labor used in each sector is given by $L_{M,t}$ and $L_{S,t}$. Crucially, labor is mobile between sectors; but the sum, L_t, is assumed to be exogenous and equal to the level of population which grows towards a finite limit L_∞:

$$L_{M,t} + L_{S,t} = L_t;\ \lim_{t\to\infty} L_t \equiv L_\infty \tag{4}$$

In our baseline simulation in Section 6, population L_t is closely matched to British history.

We use final output, *Y*, as the numeraire, normalizing its price to 1. The prices of the two goods inputs are thus related as follows:

$$\gamma^\sigma p_{M,t}^{1-\sigma} + (1-\gamma)^\sigma p_{S,t}^{1-\sigma} = 1 \tag{5}$$

and so, defining $p_t \equiv \frac{p_{M,t}}{p_{S,t}}$ and $\Gamma \equiv \frac{\gamma}{1-\gamma}$, (see appendix 1):

$$p_{S,t} = (1-\gamma)^{\frac{\sigma}{\sigma-1}}(1+\Gamma^\sigma p_t^{1-\sigma})^{\frac{1}{\sigma-1}} \text{ and } p_{M,t} = (1-\gamma)^{\frac{\sigma}{\sigma-1}}(p_t^{\sigma-1}+\Gamma^\sigma)^{\frac{1}{\sigma-1}} \qquad (6),(7)$$

The goods price ratio, p_t, is given in competitive equilibrium by:

$$p_t = \left(\frac{Y_{M,t}}{Y_{S,t}}\right)^{-\frac{1}{\sigma}} = \Gamma y_t^{-\frac{1}{\sigma}} \text{ or } y_t \equiv \frac{Y_{M,t}}{Y_{S,t}} = \Gamma^\sigma p_t^{-\sigma} \tag{8}$$

which we use later to replace p_t by the sectoral output ratio, y_t, or *vice versa*. The marginal value products and hence prices of wood and coal are respectively given by:

$$e_{M,t} = p_{M,t}\frac{\partial Y_{M,t}}{\partial \bar{E}_M} = \alpha p_{M,t}\frac{Y_{M,t}}{\bar{E}_M} \tag{9}$$

$$\bar{e}_S = p_{S,t}\frac{\partial Y_{S,t}}{\partial E_{S,t}} = \alpha p_{S,t}\frac{Y_{S,t}}{E_{S,t}} \tag{10}$$

where the coal price, $\bar{e}_S$, is assumed to be constant, as noted above. We define the energy price ratio, e_t, as the wood price relative to the coal price.

Our model thus adds two features to the model in A02:

- **Energy inputs and prices are asymmetric** ($\bar{E}_M$ vs. $E_{S,t}, e_{M,t}$ vs. $\bar{e}_S$). As shown in Equation (20) below, the optimal amount of coal depends on the number of Solow varieties, $N_{S,t}$, but wood is fixed in quantity. This means that an increase in N_S has a greater effect on output in that sector than the same increase in N_M has on output in the Malthus sector. With fixed quantities of both coal and wood, we would asymptotically have an BGP and the coal price would rise to help steer the economy back to the BGP. This asymmetry explains why our model has no BGP, and why our results are mostly more complex than those in models with BGPs. The expandability of fixed-price coal means that a transition from wood-based production to coal-based production is the

dominant development path. However, we will see that, given a high enough elasticity of substitution, a highly wood-dependent path can diverge to become ever more wood-dependent, rather than being steered back to a BGP by a falling relative coal price.

- A third production input, **labor, L, mobile** between sectors and growing exogenously over time that enhances the effect of energy asymmetry. The presence of $1-\alpha-\beta>0$ in many of our results reflects the effect of labor mobility, and if one sets $1-\alpha-\beta=0$, thus eliminating the labor inputs in (2)-(3), several of our model's early results (those not involving innovation) revert to the corresponding results in A02. Labor mobility between sectors is easier, the higher is the elasticity of substitution, σ.

Equations (9), (10) and (8) then give this expression for e_t, which along with (8) and (13) will be helpful later:

$$e_t \equiv \frac{e_{M,t}}{\bar{e}_S} = \frac{p_t y_t}{E_t} = \frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{E_t} \text{ where } E_t \equiv \frac{\bar{E}_M}{E_{S,t}} \tag{11}$$

Lastly, labor mobility between sectors results in a common wage rate w_t, equal to the marginal value product of labor in each sector:

$$w_t = p_{M,t}(1-\alpha-\beta)\frac{Y_{M,t}}{L_{M,t}} = p_{S,t}(1-\alpha-\beta)\frac{Y_{S,t}}{L_{S,t}}; \tag{12}$$

which, also using (8) and (11), gives several alternative expressions for the labor ratio l_t:

$$l_t \equiv \frac{L_{M,t}}{L_{S,t}} = \Gamma^\sigma p_t^{-(\sigma-1)} = \Gamma y_t^{\frac{\sigma-1}{\sigma}} = p_t y_t = e_t E_t \tag{13}$$

and thus from (4) the labor used in the Malthus and Solow sectors (see appendix 1):

$$L_{M,t}(p_t) = \frac{L_t \Gamma^\sigma p_t^{1-\sigma}}{1+\Gamma^\sigma p_t^{1-\sigma}} \text{ and } L_{S,t}(p_t) = \frac{L_t}{1+\Gamma^\sigma p_t^{1-\sigma}} \tag{14), (15}$$

3.2. Market for Machines, Incentives for Innovation, and Critical Values of the Elasticity of Substitution

Given the above, the first order conditions for profit maximization by competitive manufacturers of each intermediate good, $Y_i, i=M,S$, imply that the amount of each variety of machine that they demand is:

$$x_{i,t}(j) = \left(\frac{p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}}{\chi_{i,t}(j)}\right)^{\frac{1}{1-\beta}} \tag{16}$$

where $\chi_{i,t}(j)$ is the price of a machine of type j. Following A02, we set the marginal cost of manufacturing a machine at a common constant, ψ. Given our assumption that all machines are produced under a single-period patent, each machine variety is supplied by a monopolist that maximizes profit, which for variety j is given by:

$$\pi_{i,t}(j) = [\chi_{i,t}(j)-\psi]x_{i,t}(j) = p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}[x_{i,t}(j)]^{\beta} - \psi x_{i,t}(j) \tag{17}$$

Maximizing profit then yields a (privately) optimal machine price of $\chi_{i,t}^{*}(j)=\psi/\beta$, and

we set marginal cost $\psi = \beta$ so that $\chi^*_{i,t}(j) = 1$. Then, from (16), the optimal amount of each machine variety sold by each monopolist is given by:

$$x^*_{i,t}(j) = \left(p_{i,t}E^{\alpha}_{i,t}L^{1-\alpha-\beta}_{i,t}\right)^{\frac{1}{1-\beta}} \tag{18}$$

which is independent of j. Substituting this for $x_{i,t}(j)$ in the production functions (2) and (3) gives these sectoral outputs (see appendix 1 for the functions $p_{M,t}(p_t), L_{M,t}(p_t)$, etc.):

$$Y_{M,t}(p_t, N_{M,t}) = \frac{N_{M,t}}{\beta} p_{M,t}^{\frac{\beta}{1-\beta}}(p_t)\bar{E}_M^{\frac{\alpha}{1-\beta}} L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \tag{19}$$

$$Y_{S,t}(p_t, N_{S,t}) = \frac{N_{S,t}}{\beta} p_{S,t}^{\frac{\beta}{1-\beta}}(p_t)E_{S,t}^{\frac{\alpha}{1-\beta}}(p_t, N_{S,t}) L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \tag{20}$$

from which we derive two equations useful for later results.[12] One is found by dividing (19) by (20), substituting $p_t = \Gamma {y_t}^{-\frac{1}{\sigma}}$ (8) and $l_t = \Gamma y_t^{\frac{\sigma-1}{\sigma}}$ (13), and rearranging terms to get the relative goods ratio:

$$y_t(N_t, E_t) = \left(\Gamma^{1-\alpha} {E_t}^{\alpha} {N_t}^{1-\beta}\right)^{\frac{\sigma}{\theta}}, \tag{21}$$

where $N_t \equiv N_{M,t}/N_{S,t}$ is the ratio of machine varieties, which we also call the knowledge ratio, and $\theta \equiv 1 + \alpha(\sigma - 1)$ is the elasticity of substitution between the factors with labor and machines adjusting optimally. If the two energy sources were fixed in quantity, then the evolution of y_t would depend (positively and monotonically) on N_t alone.[13] The expandability of coal in our model means that this is not the case here. The other equation is for $E_{S,t}(p_t, N_{S,t})$, the optimal amount of coal in terms of the goods price ratio and coal-producing machines, which we find by substituting (20) into (10) for $\bar{e}_S$ and rearranging to get:

$$E_{S,t}(p_t, N_{S,t}) = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} {p_{S,t}}^{\frac{1}{1-\alpha-\beta}}(p_t)L_{S,t}(p_t), \tag{22}$$

which when inserted into (20) and re-arranged gives coal-sector output:

$$Y_{S,t}(p_t, N_{S,t}) = \left(\frac{N_{S,t}}{\beta}\right)^{\frac{1-\beta}{1-\alpha-\beta}} {p_{S,t}}^{\frac{\alpha+\beta}{1-\alpha-\beta}}(p_t)\left(\frac{\alpha}{\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}} L_{S,t}(p_t) \tag{23}$$

Comparing (23) to (19), we see that, as noted above, a given proportional increase in knowledge in both sectors increases output in the *S* sector more than in the *M* sector, the knowledge elasticity of the former being $(1-\beta)/(1-\alpha-\beta)$, which is larger than the unitary knowledge elasticity in the *M* sector. The reason is that any knowledge change in the *S* sector is matched by an increase in more energy inputs (see (22)), while in the *M* sector energy input necessarily stays constant.

From (17) and (18), profit per new variety, and the relative profitability of innovating in the

[12] Equations (19) and (20) are analogous to equation (15) in A02.

[13] From (11) the same will be true for the relative energy price ratio, e_t.

two sectors (both also independent of j), are therefore:

$$\pi_{i,t}(j) = [\chi_i^*(j) - \psi]x_{i,t}^*(j) = (1-\beta)\left(p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}\right)^{\frac{1}{1-\beta}}, \text{and} \tag{24}$$

$$\pi_t \equiv \frac{\pi_{M,t}(j)}{\pi_{S,t}(j)} = \left(p_t E_t^{\alpha} l_t^{1-\alpha-\beta}\right)^{\frac{1}{1-\beta}} \tag{25}$$

Equation (25) shows that the relative profitability of innovation is determined by a (goods) price effect, $p_t^{\frac{1}{1-\beta}}$, a (input) market size effect, $E_t^{\frac{\alpha}{1-\beta}}$, familiar from A02, and also a labor mobility effect, $l_t^{\frac{1-\alpha-\beta}{1-\beta}}$.[14] There is a profit incentive to innovate in the sector with the dearer good, but also in the sector that uses more inputs. As is familiar from A02, the net effect for $\sigma > 1$ is to favor innovation in the sector whose relative price is falling. As (13) shows that $l_t = p_t y_t = \Gamma^{\sigma} p_t^{-(\sigma-1)}$, labor flows to the sector whose revenue share is rising and (given $\sigma > 1$) whose relative price is falling. This effect further accentuates the incentives in favor of the sector whose relative price is falling. Comparing relative profitability (25) with relative output from the ratio of (19) and (20), we see that $\pi_t N_t = p_t y_t$, which with (8) gives:

$$\pi_t = p_t y_t N_t^{-1} = \Gamma y_t^{\frac{\sigma-1}{\sigma}} N_t^{-1} \tag{26}$$

The relative incentives for innovation are thus increasing in the ratio of the revenue shares of the intermediate goods Y_M and Y_S in final production and decreasing in the ratio of knowledge. So, there are diminishing returns to developing new varieties as the stock of varieties in a sector increases, but an incentive to innovate in the sector with growing revenue.[15] If the energy quantities were fixed then (21) shows that the relative incentives would depend on N_t alone.

3.3. Technology Innovation Process, and Asymptotic Growth

We use a variation on A02's lab equipment model, where new machine varieties generated in sector i and period t, $\Delta N_{i,t} \equiv N_{i,t} - N_{i,t-1}$, (with Δ similarly defined for all other time-dependent variables), are a function of R&D expenditure in each sector, $R_{i,t} > 0$:

$$\Delta N_{i,t} = \eta_i R_{i,t}^{\nu};\ \eta_M = \eta_S = \eta > 0, 0 < \nu < 1;\ \text{hence } \Delta N_{i,t} > 0 \tag{27}$$

We assume diminishing returns in knowledge production in each sector with respect to research expenditure as more innovating firms enter the sector and spend on R&D.[16] We also assume, as a key model feature mentioned earlier, that there is no difference in the productivity

[14] If our model had no mobile labor so that $1 - \alpha - \beta = 0$, (25) would revert to A02's (17.1).

[15] This is related to Hart's (2013) finding that relative R&D investment rates depend on the relative factor shares. We focus on this approach to describing the innovation incentives, as market size and price effects become ambiguous with the addition of the mobile labor input to the standard directed technical change model.

[16] With the time subscripts given – R&D is funded from current production – diminishing returns are needed to obtain an equilibrium. If there were constant returns to spending on R&D (and there are constant returns to knowledge in the production functions) then infinite R&D is optimal and fundable from the resulting production. We assume that new varieties generated are independent of the stock of knowledge; this is the simplest assumption compatible with a constant long-run growth rate in the Solow sector, a rate we derive at the end of Section 5.

η with which a given amount of research expenditure can produce new knowledge of either type.[17] As will be clear from (33) below, there is a scale effect of population size on the rate of innovation. However, none of our key results depend on population growth. We rearrange (27) to give the total cost of producing new varieties in each sector in a given period:

$$R_{i,t} = \left(\frac{1}{\eta}\Delta N_{i,t}\right)^{\frac{1}{\nu}} \tag{28}$$

Assuming free entry, the profit from the last variety, $\pi_{i,t}(j)$ from (24), will equal the marginal cost of innovating a new variety in a sector in a given period, $\partial R_{i,t}/\partial(\Delta N_{i,t})$, calculated from (28):[18]

$$\pi_{i,t}(j) = \left(\frac{1}{\eta}\right)^{\frac{1}{\nu}} \frac{1}{\nu} \left(\Delta N_{i,t}\right)^{\frac{1-\nu}{\nu}} \tag{29}$$

We define the relative knowledge growth rate or 'direction of technical change', n_t, as:

$$n_t \equiv \frac{\frac{\Delta N_{M,t}}{N_{M,t}}}{\frac{\Delta N_{S,t}}{N_{S,t}}} = 1 + \frac{\Delta\ln(N_t)}{\Delta\ln(N_{S,t})};\ \Delta N_t \gtreqless 0 \Leftrightarrow n_t \gtreqless 1 \tag{30}$$

Combining the above definitions with $\pi_t = \left(\frac{\Delta N_{M,t}}{\Delta N_{S,t}}\right)^{\frac{1-\nu}{\nu}}$ from (29) then shows that the direction of (proportional) technical change, n_t, is increasing in the relative profitability, π_t, of innovation and decreasing in the knowledge ratio, N_t:

$$n_t = N_t^{-1} \pi_t^{\frac{\nu}{1-\nu}} \tag{31}$$

Using this and the right-hand side of (26) we have:

$$n_t(y_t, N_t) = \Gamma^{\frac{\nu}{1-\nu}} y_t^{\left(\frac{\sigma-1}{\sigma}\right)\frac{\nu}{1-\nu}} N_t^{-\left(\frac{1}{1-\nu}\right)} \tag{32}$$

which will be the key equation for our phase-plane analysis of unbalanced growth in Section 5. The direction of technical change defined by *n*, therefore, depends on the same incentives as relative profitability. If $n_t > 1$ we will say that technical change is Malthus-directed and if $n_t < 1$it is Solow-directed. Again, with fixed energy quantities, from (21) the direction of technical change would depend only on N_t. At $n_t = 1$ there would be a BGP where *N*, *e*, and *y* (and so *p*) were constant.

Rearranging (29), dividing both sides by $N_{i,t}$ and inserting (24) for $\pi_{i,t}$ gives:

$$\frac{\Delta N_{i,t}}{N_{i,t}} = N_{i,t}^{-1}\left(\eta^{\frac{1}{\nu}}\nu\right)^{\frac{\nu}{1-\nu}} \pi_{i,t}^{\frac{\nu}{1-\nu}} = N_{i,t}^{-1}\left(\eta^{\frac{1}{\nu}}\nu(1-\beta)\right)^{\frac{\nu}{1-\nu}} \left(p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}\right)^{\frac{1}{1-\beta}\left(\frac{\nu}{1-\nu}\right)} > 0, \tag{33}$$

---

[17] We note in section 6 that allowing $\eta_M \neq \eta_S$ did little to improve our simulations' goodness of fit.

[18] Because of diminishing returns, this is an equality, so there will always be innovation in both sectors as long as both sectoral goods are produced.

showing the scale effect of population $L_{i,t}$ on innovation noted above. Next, substituting (22) into the Solow-sector version of (33) gives (see appendix 2):

$$\frac{\Delta N_{S,t}}{N_{S,t}} = \left(\eta^{\frac{1}{\nu}}\nu(1-\beta)\right)^{\frac{\nu}{1-\nu}} \left(\frac{\alpha}{\beta\bar{e}_S}\right)^{\frac{\alpha\nu}{(1-\nu)(1-\alpha-\beta)}} N_{S,t}^{\frac{\alpha}{1-\alpha-\beta}\left(\frac{\nu}{1-\nu}\right)-1} p_{S,t}^{\frac{\nu}{(1-\nu)(1-\alpha-\beta)}} L_{S,t}^{\frac{\nu}{1-\nu}} \tag{34}$$

As mentioned above, for our dynamic results in section 5 and our simulations in section 6 we impose the restriction that the economy's growth rate, and hence the Solow knowledge growth rate, $\frac{\Delta N_{S,t}}{N_{S,t}}$, is asymptotically constant. In an economy where there is an industrial revolution, production becomes ever more concentrated in the Solow, coal-using sector, so the output ratio, y_t, tends to zero. As a result, both $p_{S,t}$ and, given there is an upper bound on population, $L_{S,t}$ will tend to constants; so, a constant asymptotic growth rate requires:

$$\text{ASSUMPTION 1. } \nu = \frac{1-\alpha-\beta}{1-\beta}. \tag{35}$$

so that the exponent of $N_{S,t}$ in (34), $\frac{\alpha}{1-\alpha-\beta}\left(\frac{\nu}{1-\nu}\right) - 1$, is zero. Assumption 1 is not actually needed for most of the dynamic results in section 5, but to simplify our presentation, we use it for most of the proofs of those results.

3.4. Household

Each household supplies a unit of labor and $\bar{E}_M/L_t$ units of wood inelastically. Population is set exogenously. Consumers' income consists of wages, wood rents, and profits from the sale of machines. For simplicity we assume that producing wood does not require labor. Total consumption is $C_t = Y_t - X_t - \sum_i R_{i,t} - \bar{e}_S E_{S,t}$, where $X_t = \beta\left(\int_0^{N_{M,t}} x_{M,t}(j)dj + \int_0^{N_{S,t}} x_{S,t}(j)dj\right)$ is total expenditure on producing machines. As already noted, households are only passive consumers of final output, so we need not specify consumption any further than this.

3.5. Equilibrium

The model yields a system of three simultaneous equations for three unknowns in any period *t*: the sectoral goods price ratio. p_t, and the growth in numbers of Malthus sector (wood-using) and Solow sector (coal-using) machine varieties, $N_{M,t}$ and $N_{S,t}$. The first equation is the equilibrium between demand and supply for *y* given by (8) and the ratio of (19) and (23), respectively. The remaining two equations are given by (33) after substituting in the relevant functions that determine the rates of technical change as functions of the innovation incentives:

$$\Gamma^{\sigma} p_t^{-\sigma} = \frac{Y_{M,t}(p_t, N_{M,t})}{Y_{S,t}(p_t, N_{S,t})} \tag{36}$$

$$\frac{\Delta N_{M,t}}{N_{M,t}} = N_{M,t}^{-1}\left(\eta^{\frac{1}{\nu}}\nu(1-\beta)\right)^{\frac{\nu}{1-\nu}} \left(p_{M,t}(p_t)\bar{E}_M^{\alpha} L_{M,t}^{1-\alpha-\beta}(p_t)\right)^{\frac{\nu}{(1-\nu)(1-\beta)}} \tag{37}$$

$$\frac{\Delta N_{S,t}}{N_{S,t}} = N_{S,t}^{-1}\left(\eta^{\frac{1}{\nu}}\nu(1-\beta)\right)^{\frac{\nu}{1-\nu}}\left(p_{S,t}(p_t)E_{S,t}^{\alpha}(p_t, N_{S,t})L_{S,t}^{1-\alpha-\beta}(p_t)\right)^{\frac{\nu}{(1-\nu)(1-\beta)}} \quad (38)$$

Equations (6), (7), (14), and (15) give the explicit functional forms needed here for $p_{M,t}(p_t)$ and $L_{M,t}(p_t)$ (hence $Y_{M,t}(p_t, N_{M,t})$ via (19)), and for $p_{S,t}(p_t)$ and $L_{S,t}(p_t)$ (hence $Y_{S,t}(p_t, N_{S,t})$ via (20) and $E_{S,t}(p_t, N_{S,t})$ via (22)). Given all these functional forms and the model parameters at the start of period *t*, namely $\bar{E}_M$, $\bar{e}_S$, $N_{M,t-1}$, $N_{S,t-1}$, α, β, γ, σ, η, and L_t, we establish:

DEFINITION 1. *An equilibrium is given by the sequences of wages* (w_t)*, intermediate output prices* $(p_{M,t}, p_{S,t})$*, wood prices* $(e_{M,t})$*, coal demands* $(E_{S,t})$*, labor demands* $(L_{M,t}, L_{S,t})$*, machine demands* $(x_{M,t}, x_{S,t})$ *and expenditures on innovation* $(R_{M,t}, R_{S,t})$ *such that in each period t:* p_t*,* $N_{M,t}$ *and* $N_{S,t}$ *are simultaneously given by (36)-(38).*

4. Comparative Statics

If we fix the number of varieties $N_{M,t}$ and $N_{S,t}$ – i.e. treat them as exogenous technology parameters – the equation system consists of just the supply-demand equality (36). This section analyses this static equilibrium with exogenous technological changes, which will make it easier to understand the dynamic results in section 5. In addition to the effects of the number of varieties on the equilibrium, we also look at the effects of the price of coal, the quantity of wood, and population, and how the elasticity of substitution changes these effects.

Substituting (6), (7), (14), and (15) for prices $p_{S,t}(p_t)$, $p_{M,t}(p_t)$ and labor inputs $L_{M,t}(p_t)$, $L_{S,t}(p_t)$ into the ratio of (19) and (23), we obtain the relative supply of the two outputs on the right-hand side of (36), which for clarity we will label here as y_t^s:

$$y_t^s(\bar{N}_{M,t}, \bar{N}_{S,t}, p_t) =$$

$$\phi \bar{N}_{M,t} \bar{N}_{S,t}^{-\left(\frac{1-\beta}{1-\alpha-\beta}\right)} \bar{E}_M^{\frac{\alpha}{1-\beta}} \left(\frac{\alpha}{\beta \bar{e}_S}\right)^{-\left(\frac{\alpha}{1-\alpha-\beta}\right)} p_t^{\frac{\beta-(\sigma-1)(1-\alpha-\beta)}{1-\beta}} \quad (39)$$

$$\left(1+\Gamma^{\sigma} p_t^{-(\sigma-1)}\right)^{\left(\frac{\alpha((\sigma-1)(1-\alpha-\beta)-1)}{(\sigma-1)(1-\beta)(1-\alpha-\beta)}\right)} L_t^{-\left(\frac{\alpha}{1-\beta}\right)}$$

where $\phi = (1-\gamma)^{-\frac{\alpha\sigma}{(\sigma-1)(1-\beta)(1-\alpha-\beta)}} \Gamma^{\frac{\sigma(1-\alpha-\beta)}{1-\beta}}$. Figure 5 plots the relative supply (39) and relative demand, $y_t^d = \Gamma^{\sigma} p_t^{-\sigma}$ from (8), using our baseline parameters from section 6 and the values of other variables in 1560 in panel b, and with the same parameters but with $\sigma = 1.85$ instead of 4 in panel a. As in our simulation, figure 5 uses a normalized production function with the base period of 1560, so changes in σ do not change the equilibrium price and quantity but only the slopes of the supply and demand curves through the equilibrium point.

By computing the supply elasticity, $\rho(\sigma) = \frac{\partial \ln y_t^s}{\partial \ln p_t}$ from (39), which we do in appendix 3, we can show that the relative supply curve is upward sloping for $\sigma < 1 + \frac{\beta}{1-\alpha-\beta}$ (roughly 1.43 and less for our parameterization in section 6), downward sloping for $\sigma > \frac{1}{1-\alpha-\beta}$ (about 1.9 and greater here) of σ, and backward bending for an intermediate range of σ as in figure 5a. Intuitively, the downward slope of the relative supply curve for higher values of σ is due to the

intersectoral labor mobility effect increasing with σ. As equations (19) and (23) show, outputs Y_M and Y_S are rising in their own prices as we would expect. However, sectoral labor allocation depends on the endogenous wage, and equations (14)-(15) show that for $\sigma > 1$, each sector's labor use declines with its own sector output price, and as its sector output price rises, the output price of the other sector falls, strengthening this effect. The greater σ is, the more this labor mobility effect acts against the positive own price effects in (19) and (23); hence for higher values of σ, outputs are falling in their own prices, and rising *p* means falling *y*.

From (39), the parameters shift the relative supply curve as follows:

$$\frac{\partial \ln y_t^s}{\partial \ln \bar{N}_{M,t}} = 1, \frac{\partial \ln y_t^s}{\partial \ln \bar{E}_M} = \frac{\alpha}{1-\beta}, \frac{\partial \ln y_t^s}{\partial \ln \bar{e}_S} = \frac{\alpha}{1-\alpha-\beta}, \frac{\partial \ln y_t^s}{\partial \ln \bar{N}_{S,t}} = -\frac{1-\beta}{1-\alpha-\beta}, \frac{\partial \ln y_t^s}{\partial \ln L_t} = -\frac{\alpha}{1-\beta} \quad (40)$$

The signs of the first four derivatives are intuitive. For example, one would expect scarcer wood or cheaper coal to spur industrialization. Increased population shifts the relative supply to the left because it increases the optimal amount of coal but obviously does not affect the quantity of wood.

The *equilibrium* changes in relative output *y* also depend, of course, on both the demand elasticity, which from (8) is -σ, and the supply elasticity, $\rho(\sigma)$. Using a standard comparative static result, the equilibrium response of relative output to a shift in the relative supply function is given by

$$d\ln y = \left(\frac{\sigma}{\rho(\sigma)+\sigma}\right) d\ln y^s \quad (41)$$

where $d\ln y^s$ is an exogenous *shift* (at constant *p*) in the relative output supply curve, caused for example by a change in *N*, the knowledge ratio. In appendix 4 we demonstrate the following proposition:

PROPOSITION 1. *Any of fewer wood-using varieties, $\bar{N}_{M,t}$, a lower wood quantity, $\bar{E}_M$, a lower coal price $\bar{e}_S$, more coal-using varieties, $\bar{N}_{S,t}$ or higher population, L_t, move the economy towards locally lower y_t (i.e. higher industrialization):*

$$\partial y_t/\partial \bar{N}_{M,t}, \partial y_t/\partial \bar{E}_M, \partial y_t/\partial \bar{e}_S > 0; \; \partial y_t/\partial \bar{N}_{S,t}, \partial y_t/\partial L_t < 0 \quad (42)$$

Additionally, an equiproportional increase in $\bar{N}_{M,t}$ and $\bar{N}_{S,t}$, i.e. $\Delta ln(\bar{N}_{M,t}) = \Delta ln(\bar{N}_{S,t}) > 0$, hence $\Delta ln(\bar{N}_t) = 0$, results in lower y_t.

Proof. See appendix 4.[19]

For our dynamic analysis in Section 5, we would like to know how the elasticity of substitution, σ, affects these comparative statics. To provide some intuition, we look at how σ affects the price elasticity of the relative supply (39) and demand (8) functions, and how in turn this affects the comparative statics. It turns out (see appendix 5 for details) that for most parameter and price values, including all those relevant to the Pre-industrial Stagnation

[19] Throughout our analysis, we approximate what are formally differences in discrete time ($\Delta y_t, \Delta N_t$, etc.) as differentials in continuous time (dy, dN, etc). Many of our equals signs (=) should, therefore, strictly be replaced by approximately equals signs (≈); but our many simulations (mostly not reported in section 6) have confirmed that the formally approximate analytic results thus found here hold true numerically. So, to avoid complexity which adds no insights, we have used only equals signs.

behavior discussed in section 5.2 below, $\frac{\rho(\sigma)}{\sigma}$ decreases, hence $\frac{\sigma}{\rho(\sigma)+\sigma}$ increases, with increasing σ. This is illustrated by the move from figure 5a to 5b, where the supply elasticity $\rho(\sigma)$ goes from near zero to rather negative as σ increases. Hence from (41), when $\partial \ln y_t/\partial \ln N_t$ is positive, it increases with the elasticity of substitution, σ, a result we return to in Section 5.2 below.

5. Dynamic Results

5.1 Introduction

We now analyze the dynamics of the complete system (36)-(38) where growth in the numbers of machine varieties is determined by the incentives for innovation. We describe the paths that the economy can take using the labels Industrial Revolution, Modern Economic Growth and Pre-industrial Stagnation, seen in figures 6a-7b (which we introduce shortly), and formally defined as:

DEFINITION 2. *A development path of the model undergoes an Industrial Revolution if* $\Delta N_t < 0$ *and* $\Delta y_t < 0$ *forever after some t on the path, so that Solow machine varieties and goods output are rising relative to Malthus varieties and output, with* $N_t \to 0$ *and* $y_t \to 0$ *as* $t \to \infty$. *Modern Economic Growth occurs on an Industrial Revolution development path if the energy price ratio,* e_t, *is falling along the path. Pre-industrial Stagnation occurs on a path if* $\Delta N_t > 0, \Delta y_t > 0$ *initially and forever, with* $N_t \to \infty$ *and* $y_t \to \infty$ *as* $t \to \infty$, *so it never undergoes an Industrial Revolution.*

We establish three dynamic propositions, arranged in order of importance and in what is effectively the UK's historical sequence, and dependent on which of these three ranges of values that substitutability σ falls into:

DEFINITION 3. *We refer to substitutability as being Low if* $1 < \sigma < \tilde{\sigma}$, *Medium if* $\tilde{\sigma} < \sigma < \sigma^{\dagger}$ *as in figures 6a and 7a, and High if* $\sigma > \sigma^{\dagger}$ *as in figures 6b and 7b, where:*

$$\tilde{\sigma} \equiv 1 + \frac{1}{1-\alpha-\beta} < \sigma^{\dagger} \equiv 1 + \frac{1-\beta}{(1-\alpha-\beta)^2} \tag{43}$$

Proposition 2 (sec. 5.2) will show how, given High substitutability, a zone of Pre-industrial Stagnation will exist, in which the economy will remain if its initial conditions start it in that zone (fig. 6b). Proposition 3 (sec. 5.3) will show how, given Medium or High substitutability, Industrial Revolution paths will end up with Modern Economic Growth (figs. 7a and 7b). Section 5.4 explains why High substitutability is needed for Pre-industrial Stagnation but only Medium substitutability is needed for Modern Economic Growth. Finally, proposition 4 (sec. 5.5) will show how Assumption 1 means the economy's asymptotic growth rate (of total output per person) is a positive constant on an Industrial Revolution path, but zero on a Pre-industrial Stagnation path.

5.2. Industrial Revolution or Pre-industrial Stagnation?

Figures 6a and 6b – phase diagrams in (*y,N*)-space for Medium and High substitutability – plot the dynamic evolution of the system as represented by (32): $(n_t(y_t, N_t) =$

$\Gamma^{\frac{\nu}{1-\nu}} y_t^{\left(\frac{\sigma-1}{\sigma}\right)\frac{\nu}{1-\nu}} N_t^{-\left(\frac{1}{1-\nu}\right)}$). Together, these figures illustrate most of our dynamic results and will be explained below.[20] All development paths are unbalanced, asymptotically approaching either the origin (relative output and machine varieties, $y_t, N_t \rightarrow 0$), or (in fig. 6b only) infinity ($y_t, N_t \rightarrow \infty$), with no paths approaching fixed values of the ratios y and N characteristic of balanced growth. Figure 6a shows that for Medium substitutability, all development paths converge to the origin, with output and machine varieties in the Solow sector growing relative to those in the Malthus sector. Initially, though, the economy follows a path where Malthus output and machine varieties grow faster than Solow ones. Figure 6b shows that when the elasticity of substitution is High, if the economy starts out in the Pre-industrial Stagnation Zone (whose existence will be demonstrated in proposition 2), where relatively high abundance of wood means that relative output, y, is high given the knowledge ratio, N, then the economy will never cross the separatrix into the Industrial Revolution zone.

Intuitively, the reason why all development paths in figure 6a and most paths in figure 6b eventually turn and undergo an industrial revolution (falling y_t) is because coal use rises along these development paths while wood use is fixed. On the arrowed path in figure 6a, initially knowledge growth is more rapid in the Malthus sector ($n_t > 1$), so y_t rises. The rise in coal use, however, boosts the profit incentive to research in the Solow sector (lowers π_t in (24)) by enough to lower n_t hence slow the rise of y_t. In time, coal use rises (E_t falls) fast enough to turn around the direction of development (see (21)), despite the direction of technical change still being in the Malthus direction: the path crosses the $\Delta y_t = 0$ isocline even though N_t is still rising. By contrast, in the Pre-industrial Stagnation Zone in figure 6b, as we will show, coal use is falling, so development paths never turn around but instead diverge from Industrial Revolution paths.

From (32), the $\Delta N_t = 0$ isocline in figures 6a and 6b and the sign of ΔN_t above and below the isocline are given by:

$$\Delta N_t \gtreqless 0 \Leftrightarrow N_t \lesseqgtr \Gamma^{\nu} y_t^{\left(\frac{\sigma-1}{\sigma}\right)\nu} \tag{44}$$

The isocline is upward sloping because relative market size as measured by the revenue share ratio py ($= y^{\frac{\sigma-1}{\sigma}}$) has to be larger for an increased knowledge ratio N to offset the diminishing returns to knowledge. As stated in proposition 1, if knowledge in the Malthus and Solow sectors grows at the same proportional rate ($n_t = 1$) then relative output, y_t, is falling. So, the $\Delta y_t = 0$ isocline must be below the $\Delta N = 0$ isocline, giving rise to unbalanced growth.[21] Formally, appendix 6 proves that:

$$\Delta y_t \gtreqless 0 \;\Leftrightarrow\; n_t \gtreqless \frac{1-\beta}{1-\alpha-\beta} > 1 \;\Leftrightarrow\; N_t \lesseqgtr \left(\frac{1-\alpha-\beta}{1-\beta}\right)^{1-\nu} \Gamma^{\nu} y_t^{\left(\frac{\sigma-1}{\sigma}\right)\nu} \tag{45}$$

[20] Unlike phase diagrams for standard dynamic optimization models, where a control variable like consumption can jump in response to a shock, but a state variable like capital is pre-determined by history, in our phase diagrams not only is the relative knowledge stock N pre-determined, but also relative output y is some sense pre-determined by equation (36).

[21] Equation (21) shows that with fixed energy quantities, the $\Delta y_t = 0$ and $\Delta N_t = 0$ isoclines would collapse to a single common curve.

We now prove the key property of the High substitutability case: that the (y,N) phase-space is separated into a lower zone of Pre-industrial Stagnation development paths and an upper zone of Industrial Revolution paths with a separatrix between them. An analytical proof exists only given the extra, counterfactual assumption of constant population, but we then discuss below the extension by continuity to the historical case of population growth.

PROPOSITION 2. *(a) Given a High elasticity of substitution ($\sigma > \sigma^{\dagger}$) and constant population, there is a monotonic increasing separatrix in (y, N)-space lying strictly below the $\Delta y_t = 0$ isocline (45); all paths below this separatrix exhibit Pre-industrial Stagnation, with eventually $\Delta n_t > 0$ forever, and all paths above it are Industrial Revolution paths, as in figure 6b. (b) Given a Medium or Low elasticity of substitution ($\sigma < \sigma^{\dagger}$), all development paths undergo an Industrial Revolution (i.e., Pre-industrial Stagnation is not possible).*

Proof. See appendix 7.

With *population growth* ($\Delta\ln(L_t) > 0$), no analytic proof of part (a) is possible. But by continuity, proposition 2(a) holds for at least some small level of population growth,[22] and our numerical simulations found that it does indeed hold for historical British population growth, and moreover for a wide range of variants on our baseline simulation.

The following explains intuitively why, given enough initial relative wood abundance, High σ gives rise to Pre-industrial Stagnation. Taking logs and first differences of (32) and using Assumption 1 (35), we have:

$$\Delta\ln n_t = \frac{1-\alpha-\beta}{\alpha}\left(\frac{\sigma-1}{\sigma}\right)\Delta\ln y_t - \frac{1-\beta}{\alpha}\Delta\ln N_t \tag{46}$$

In order to get Pre-industrial Stagnation, we need $\Delta\ln n_t > 0$ so that technical change is forever increasingly Malthus-directed. The higher σ is, then, as from (13) $p_t y_t = \Gamma y_t^{\left(\frac{\sigma-1}{\sigma}\right)}$, the more the relative revenue share ratio, $p_t y_t$, increases for a given increase in y_t, increasing the incentive to innovate in Malthus technology rather than Solow technology. Both y and N are increasing in the relevant zone of the phase diagram below the $\Delta y_t = 0$ isocline (fig. 6b). As discussed in section 4, y is driven by changes in N so that here $\partial\ln y_t/\partial\ln N_t > 0$. When the indirect effect of the relative growth of Malthus knowledge, $\frac{1-\alpha-\beta}{\alpha}\left(\frac{\sigma-1}{\sigma}\right)\frac{\partial\ln y_t}{\partial\ln N_t}\Delta\ln N_t$ in (46), outweighs its direct effect, $-\frac{1-\beta}{\alpha}\Delta\ln N_t$, the economy will remain in Pre-industrial Stagnation. In this zone, growth in the ratio of the sectors' revenue shares will have a greater effect on incentivizing Malthus-sector innovation than the effect of diminishing returns to relative knowledge accumulation, creating a self-reinforcing spiral.

As discussed at the end of section 4, $\partial\ln y_t/\partial\ln N_t$ increases with σ for relevant values of the parameters and relative supply, y_t. Hence for values of σ above a minimum bound there will be a zone of Pre-industrial Stagnation in part of the y-N space below the $\Delta y_t = 0$ isocline

[22] All key steps in proposition 2(a)'s proof use inequalities (rather than equalities) that would remain true for some small level of population growth, since all our functions are continuously differentiable. For what growth level they would remain true, however, our theory cannot say.

where n_t is indeed rising. The end of appendix 7 shows how we can derive the bounding value of the elasticity of substitution, $\sigma^{\dagger}$, from further analysis of equation (46).

5.3 Modern Economic Growth and Energy Prices

In this section we show how for Medium and High substitutability the relative price of wood to coal eventually falls even though the relative use of coal is rising, in a phase we call Modern Economic Growth.

Using (13) to change relative energy quantities, E_t, in (21) to relative energy prices, e_t, and rearranging gives the conversion from the (*y,N*)-plane to the (*e,N*)-plane:

$$y_t = \Gamma^{\sigma} e_t^{-\alpha\sigma} N_t^{(1-\beta)\sigma}, \tag{47}$$

which we put into the (*y,N*) phase-plane equation (32) and, using Assumption 1, obtain the (*e,N*) phase-plane equation:

$$n_t(e_t, N_t) = \Gamma^{\frac{\sigma(1-\alpha-\beta)}{\alpha}} e_t^{-(\sigma-1)(1-\alpha-\beta)} N_t^{\frac{(1-\beta)(1-\alpha-\beta)(\sigma-\tilde{\sigma})}{\alpha}}. \tag{48}$$

which is plotted in figures 7a and 7b for the Medium ($\tilde{\sigma} < \sigma < \sigma^{\dagger}$) and High ($\sigma > \sigma^{\dagger}$) substitutability cases. In figure 7a, all development paths eventually converge to the origin as in figure 6a. In figure 7b, most paths converge to the origin and Modern Economic Growth, but the relative price of wood rises forever along Pre-industrial Stagnation paths. Equation (48) shows, that for $\sigma > 1$, for a given knowledge ratio, *N*, higher relative wood scarcity discourages wood-directed innovation and encourages coal directed innovation.

From (48), the isocline $\Delta N_t = 0$ in (*e,N*)-space and the signs of N_t above and below it are given by:

$$n_t \gtreqless 1 \Leftrightarrow \Delta N_t \gtreqless 0 \Leftrightarrow N_t \gtreqless \Gamma^{\frac{-\sigma}{(1-\beta)(\sigma-\tilde{\sigma})}} e_t^{\frac{\alpha(\sigma-1)}{(1-\beta)(\sigma-\tilde{\sigma})}}. \tag{49}$$

Hence the $\Delta N_t = 0$ isocline is rising in (*e,N*)-space if $\sigma > \tilde{\sigma}$; and concave (convex) if $\frac{\alpha(\sigma-1)}{(1-\beta)(\sigma-\tilde{\sigma})} > (<) 1$, which from (43) implies $\sigma > (<)\ \sigma^{\dagger}$.

We now prove formally, and illustrate intuitively, the existence of the $\Delta e_t = 0$ isocline in figures 7a and 7b, the phase diagrams in (*e,N*)-space, above which the relative price of wood to coal e_t rises, and below which it falls. The falling relative price of wood is the ultimate effect of self-reinforcing technical change. This is an example of strong (relative) biased technical change (Acemoglu, 2002, 2007), where the relative price of coal increases alongside increasing coal use, which happens in all of the "Modern Economic Growth" zone (def. 2) shown in figures 7a and 7b. Formally, we can state:

PROPOSITION 3. *(a) Given Medium or High substitutability ($\sigma > \tilde{\sigma}$) and constant population, an upward-sloping isocline $\Delta e_t = 0$ occurs below the $\Delta N_t = 0$ isocline in (e, N)-space, with $\Delta e_t > 0$ above the former isocline and $\Delta e_t < 0$ (Modern Economic Growth) below it, as in figures 7a and 7b. (b) There is Strong Relative Bias throughout the Modern Economic Growth zone, i.e. relative coal use ($E_{S,t}/\bar{E}_M$) is rising even though its relative price ($\bar{e}_S/e_{M,t}$) is also rising.*

Proof. See appendix 8.

Strong relative biased technical change occurs when the shift in relative supply of the inputs due to technical change is relatively small compared to the shift in their demand curve, so that the relative price of the inputs rises (falls) as their relative supply rises (falls). The relative inverse demand function for the two energy inputs is given by substituting (21) into (11) and rearranging:

$$e_t(E_t, N_t) = \Gamma^{\sigma/\theta} E_t^{-1/\theta} N_t^{(1-\beta)(\sigma-1)/\theta} \tag{50}$$

This demand curve is downward sloping; and given $\sigma > 1$, increases in N – relatively wood-directed technical change – shift the demand curve up and *vice versa*, so that wood-directed technical change is also wood-biased. In other words, for constant *E*, the relative price of wood to coal, e, increases. To determine where on the relative demand curve the equilibrium point is, we use the optimal coal quantity (22), which plays the role of "supply" here.

In figures 7a and 7b, below the $\Delta N = 0$ isocline, growth is coal-biased and technical change is shifting the relative energy demand curve (50) down over time. From (50), the higher σ is, the greater the exponent of *N* is,[23] hence the more that coal-biased technical change (falling *N*) shifts the relative energy demand curve down. Intuitively, the greater the flexibility of production, the more the relative marginal products of the factors move due to changes in the ratios of machine varieties and labor (with the quantity of the factors, *E*, held constant). However, *E* is falling because coal use is rising over time (according to (22)), which counteracts the effect of falling *N*.

If σ is great enough, then the shift down of the relative energy demand curve eventually becomes more rapid than the decline in *E* due to the expansion of coal use (wood use is fixed) so that there is a strong relative bias of technical change towards coal below the $\Delta e = 0$ isocline. This takes time to eventuate because initially as we cross the $\Delta N = 0$ isocline, technical change is still not so biased and does not shift the relative energy demand curve (50) by much compared to the decline in *E*. So, at first *e* continues to rise and the $\Delta e = 0$ isocline is below the $\Delta N = 0$ isocline.

Taking this process to the limit on the far left of figures 7a and 7b in the late stages of Modern Economic Growth as $e \to 0$ and, therefore, $p \to \infty$. As we explained when introducing assumption 1, both the price of Solow-sector output, $p_{S,t}$, and the quantity of labor in the sector, $L_{S,t}$, will tend to constants, so from (22) the growth rate of coal use, $\Delta \ln E_{S,t}, \to \frac{1-\beta}{1-\alpha-\beta} \Delta \ln N_{S,t}$. Also, $\Delta N_{M,t} \to 0$, so the main driver of relative knowledge growth, $\Delta \ln N_t$, is $\Delta \ln N_{S,t}$.[24] The numerator of the exponent of N_t in (50) is $(1-\beta)(\sigma-1)$. If this is larger than $\frac{1-\beta}{1-\alpha-\beta}$, then "supply" of coal relative to wood will increase slower than the inverse relative demand curve shifts, resulting in strong bias to coal. This is the case for $\sigma > 1 + \frac{1}{1-\alpha-\beta} \equiv \tilde{\sigma}$, i.e. Medium substitutability, hence explaining why this is the threshold value needed for proposition 4 to hold.

---

[23] Recall that $\theta \equiv 1 + \alpha(\sigma - 1)$, so $(d/d\sigma)[(\sigma-1)/\theta] = 1/\theta^2 > 0$.

[24] Substituting $p_{M,t}(p_t)$ (7) and $L_{M,t}(p_t)$ (14) into equation (37) readily shows that $\Delta N_{M,t}$ declines towards zero as $y \to 0$, provided $\sigma > \tilde{\sigma}$.

5.4 Why Does Pre-Industrial Stagnation Require a Greater Elasticity of Substitution than Modern Economic Growth?

Equation (48) above readily shows that if $\sigma > \tilde{\sigma}$ it is possible for *N*, *e*, and *n* to all decline together; if so, there is then strong bias to coal and increasingly Solow-directed technical change. Proposition 3 showed that, indeed, strong bias to coal can occur when $\sigma > \tilde{\sigma}$. Furthermore, if we are not in Pre-industrial Stagnation, *n* must be falling. Equation (48) also suggests that the reverse should be possible, but proposition 2 shows that only if $\sigma > \sigma^{\dagger}$ can Pre-industrial Stagnation actually occur. In appendix 9, we show that strong bias to wood where *E* and *e* are rising together, and thus coal use is declining, is possible for $\sigma > \tilde{\sigma}$ for some part of the phase plane below the $\Delta e = 0$ isocline in figures 7a and 7b. However, it turns out that unless $\sigma > \sigma^{\dagger}$, these paths eventually will have rising rather than falling coal use and proceed to an industrial revolution. Technical change on these paths is initially Malthus-directed but becomes less and less so. This is essentially due to the asymmetric energy supplies that do not allow for the expansion of wood use.

Next, we show that there must be strong equilibrium bias for Pre-industrial Stagnation to occur, using the following equation, derived in appendix 9 by combining the energy demand curve (50) and the *(e,N)* phase-plane equation (48):

$$n_t(e_t, E_t) = \Gamma^{\frac{\sigma}{\alpha(\sigma-1)}} e_t^{\frac{(\sigma-1)(1-2\alpha-\beta)-1}{\alpha(\sigma-1)}} E_t^{\frac{(\sigma-1)(1-\alpha-\beta)-1}{\alpha(\sigma-1)}} \tag{51}$$

This equation expresses the direction of technical change, and hence also the incentives for innovation, in terms of an energy price effect, $e_t^{\frac{(\sigma-1)(1-2\alpha-\beta)-1}{\alpha(\sigma-1)}}$, and an energy market size effect, $E_t^{\frac{(\sigma-1)(1-\alpha-\beta)-1}{\alpha(\sigma-1)}}$. For $\sigma > \tilde{\sigma}$ this market size effect is positive. The price effect is negative – a more expensive energy resource disincentivizes innovation in a sector – unless $\sigma > 1 + \frac{1}{1-2\alpha-\beta} > \sigma^{\dagger}$.[25] So, as long as $\sigma < 1 + \frac{1}{1-2\alpha-\beta}$, we have to have strongly wood-biased technical change, where *E* and not just *e* is increasing, in order to get Pre-industrial Stagnation. Therefore, the elasticity that allows for Pre-industrial Stagnation must be larger than the one that allows for strongly biased technical change. The ultimate reason for the difference in powers of *E* and *e* in (51) is the presence of *E* in (21), which is one source of (48); for as we noted after (21), if both energy sources were fixed, *E* would not appear there and *y* would depend only on *N*. This shows how the difference between $\sigma^{\dagger}$ and $\tilde{\sigma}$ is due to the energy supply asymmetry.

[25] To understand why the sign of the energy price effect flips as σ gets even greater, we see from (50) that constant *E* means that *N* must increase to increase *e*. But as we explained in section 5.2, it also increases *y* and through (32) *n*. So, these other effects become more powerful than the disincentive of more expensive resources. A similar story explains why the market size effect in (51) is negative for lower values of σ.

5.5 Asymptotic Growth Rates under High Substitutability

We can formally show the rate of growth is asymptotically zero under Pre-industrial Stagnation, while, by assumption, the growth rate on an Industrial Revolution path is asymptotically constant. Given assumption 1, the Malthus-sector version of (33) is

$$\frac{\Delta N_{M,t}}{N_{M,t}} = \lambda_M N_{M,t}^{-1}\left(p_{M,t} L_{M,t}^{1-\alpha-\beta}\right)^{\frac{1-\alpha-\beta}{\alpha(1-\beta)}} \text{ where } \lambda_M \equiv \left[\eta^{\frac{1-\beta}{1-\alpha-\beta}}(1-\alpha-\beta)\bar{E}_M^{\frac{\alpha}{1-\beta}}\right]^{\frac{1-\alpha-\beta}{\alpha}} \quad (52)$$

while (34) becomes:

$$\frac{\Delta N_{S,t}}{N_{S,t}} = \lambda_S\, p_{S,t}^{\frac{1}{\alpha}}\, L_{S,t}^{\frac{1-\alpha-\beta}{\alpha}} \text{ where } \lambda_S \equiv \eta^{\frac{1-\beta}{\alpha}}(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}}\left(\frac{\alpha}{\beta \bar{e}_S}\right) \quad (53)$$

which then allows us to show:

PROPOSITION 4. *Given the upper bound on population,* L_∞*, economic growth (i.e. growth of* Y_t/L_t*, final output per capita) is asymptotically zero under Pre-industrial Stagnation, when this path exists; and asymptotically constant under an Industrial Revolution:*

$$\lim_{t\to\infty;PS} \frac{\Delta(Y_t/L_t)}{Y_t/L_t} = 0 \quad (54)$$

$$\lim_{t\to\infty;IR} \frac{\Delta(Y_t/L_t)}{Y_t/L_t} = \left(\frac{1-\beta}{1-\alpha-\beta}\right)\lambda_S(1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{1}{\alpha}\right)} L_\infty^{\frac{1-\alpha-\beta}{\alpha}} \quad (55)$$

Proof. See appendix 10.

This asymmetry between sectors again stems ultimately from the expandable coal supply, $E_{S,t}$, in the Solow-sector production function (20), compared to the non-expandable wood supply, $\bar{E}_M$, in the Malthus-sector version (19) and from our assumption in (4) of an asymptotic limit to population.[26] The growth rate in (55) increases with the elasticity of substitution, as predicted by de la Grandville (1989).

6. Simulations

In this section, we use a Matlab program to find numerical solutions for the dynamic equilibrium defined by definition 1. We first show how a baseline simulation of our model, fitted to the stylized facts of the British Industrial Revolution using reasonable parameter values, illustrates proposition 3. We then present counterfactual simulations which illustrate the comparative statics presented in section 4, that the Industrial Revolution would have been delayed by any of: more abundant wood, a higher coal price, less initial Solow knowledge, or less population growth. We also find that the Industrial Revolution would have been delayed by a higher elasticity of substitution between sectoral goods, consistent with proposition 2(a) that Pre-industrial Stagnation is possible only for high enough substitutability.

[26] It can be shown that with asymptotically growing population, and also a small enough energy share ($\alpha < (1-\beta)/2$), positive growth would remain possible under Pre-industrial Stagnation. We thank a referee for this point.

6.1. Population Calibration

For our historical baseline and counterfactual simulation scenarios we provide the exogenous population input parametrically. We refitted Marchetti *et al.*'s (1996) bilogistic function model using Broadberry *et al.*'s (2015) data for the population of the United Kingdom at 20-year intervals through 2000, resulting in the following fit:

$$S_\tau = \frac{9.7}{1 + exp\left(-\frac{\ln(81)}{267}(\tau - 1530)\right)} + \frac{47.4}{1 + exp\left(-\frac{\ln(81)}{171}(\tau - 1870)\right)} \to 57.1 \equiv S_\infty \text{ as } \tau \to \infty \quad (56)$$

where τ is the calendar year and population S_τ is measured in millions. Then we assume that the total (normalized) labor force is given by $L_t \equiv S_t/S_1$ (so $L_\infty \equiv S_\infty/S_1$), where time *t* counts 20-year periods from $t = 1$ in 1560, the first year of Warde's (2007) energy data, to $t = 18$ in 1900, so that $t = (\tau - 1540)/20$.

6.2. Baseline Simulation

The simulations use a normalized CES production function:

$$\frac{Y_t}{Y_b} = \left[\gamma\left(\frac{Y_{M,t}}{Y_{M,b}}\right)^{\frac{\sigma-1}{\sigma}} + (1-\gamma)\left(\frac{Y_{S,t}}{Y_{S,b}}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \quad (57)$$

where Y_b is final output and $Y_{M,b}$ and $Y_{S,b}$ are the sectoral outputs in the first period of the Baseline simulation, γ is the share parameter corresponding to normalized sectoral outputs $Y_{M,t}/Y_{M,b}$ and $Y_{S,t}/Y_{S,b}$,[27] whose prices are now defined in terms of normalized final output Y_t/Y_b, and other equations are modified accordingly. This ensures that comparisons involving changes in the elasticity of substitution are meaningful (Klump *et al.*, 2012).[28]

Our full set of Baseline parameters is shown in table 1. We take the cost share of energy in 1800 in Britain to be around 25% not including human and animal power (Gentvilaite *et al.*, 2015), so we set the energy output elasticity to $\alpha = 0.25$. We set the output elasticity of machines to $\beta = 0.225$ based on table 13 in Clark (2010). We normalize the quantity of wood, $\bar{E}_M$, and the stock of machine varieties in the Solow sector in 1540 ($t = 0$), $N_{S,0}$., to 1. We set σ = 4, close to Kander and Stern's (2014) estimate and above $\sigma^\dagger$ (= 3.81 given our choices of α and β), that is, 'High'.

The remaining parameters are $N_{M,0}$, η, $\bar{e}_S$, and γ. We optimize these by minimizing the sum of squared proportional deviations from six Stylized Facts: two based on the initial state in Britain in 1560, and four based on the change in the variables over its Industrial Revolution.[29]

---

[27] In general, normalization results in γ in (1) and γ in (57) having different numerical values.

[28] As under normalization $p_1 = \Gamma$, the equations for the first period of the Baseline simulation require solving for only $N_{M,1}$ and $N_{S,1}$. The values of $Y_{M,1}$ and $Y_{S,1}$ are then treated as parameters in the remaining periods of the Baseline and for all periods of the counterfactual simulations.

[29] As part of these optimizations, at first, we allowed different innovation productivities η_M and η_S, but found that doing so gave very little reduction in the sum of squared deviations. This strengthens our case for using a common η in the innovation equation (27), to avoid unhelpful theoretical complexity.

Using calendar year time subscripts, the chosen stylized facts and proportional deviations are respectively:

1. The price of wood is double the price of coal in 1560 (Allen, 2009): $\ln\left(\frac{e_{M,1560}}{e_{S,1560}}\right) - \ln(2)$.
2. Coal use is 30% of wood use in 1560 (Warde, 2007): $\ln\left(\frac{E_{S,1560}}{\bar{E}_M}\right) - \ln(0.3)$.
3. The price of wood doubles from 1560 to its peak (Allen, 2009): $\ln\left(\frac{max(e_{M,t})}{e_{M,1560}}\right) - \ln(2)$.
4. Energy intensity doubles from 1560 to 1900 (Warde, 2007; Broadberry *et al.*, 2015): $\ln\left(\frac{(E/Y)_{1900}}{(E/Y)_{1560}}\right) - \ln(2)$.[30]
5. Output per capita rises 5.4-fold from 1560 to 1900 (Broadberry *et al.*, 2015): $\ln\left(\frac{(Y/L)_{1900}}{(Y/L)_{1560}}\right) - \ln(5.4)$.
6. Output per capita doubles from 1560 to 1800 (Broadberry *et al.*, 2015): $\ln\left(\frac{(Y/L)_{1800}}{(Y/L)_{1560}}\right) - \ln(2)$.

Figure 8a graphs our Baseline simulation results over time for four ratios: the Malthus sector's share of labor input, $L_{M,t}/L_t$, the wood/coal relative price, e_t, coal use relative to wood use, $E_{S,t}$, and output per capita, Y_t/L_t. Coal use is converted to logarithms because its overall growth is so great. Also shown are the corresponding data for the latter three variables based on that presented in figures 1 to 3. The solid lines show the simulated data, while the round, triangular, and square markers show the historical values for the wood/coal price, ln (coal/wood) quantity ratios, and output per capita.[31] Simulated results are broadly comparable to the historical data and illustrate both parts of proposition 3, extended to a growing population. However, both the peak in the simulated wood/coal price and the acceleration in economic growth come somewhat later than they do historically. This could be because of factors outside our simple model, such as the growth of trade and the reform of agriculture, which might have contributed to growth in output per capita in the 17th and 18th Centuries estimated by Broadberry *et al.* (2015). Our simulated final output better fits Clark's (2010) data, which shows only a 22% increase from 1560 to 1800. Though there is technological change, the exogenous growth of population is sufficient to reduce output per capita in this period in our simulation. The Malthus sector's share of labor falls from 85% in 1560 to 4% in 1900, which seems reasonable. Coal use increases 250-fold by 1900 in our simulation, which is less than in reality. The model fits Stylized Facts 1 to 5 very well, but Fact 6 less well. The latter is perhaps not surprising given how simplified the model is – for example assuming equal

[30] This reflects the increase in total energy intensity in figure 4. We tried instead using a ratio of 4 in our optimization, to reflect the increase in firewood and coal energy intensity, but this resulted in a much poorer fit to the other stylized facts.

[31] Note that this figure and figure 8a present ratios of the prices and quantities of wood and coal, while figures 2 and 3 present the original series.

parameter values in each sector – and how much uncertainty there is about the values of those parameters.[32]

6.3. Counterfactual Simulations

In figs. 8b-h we simulate the following seven counterfactual scenarios, to highlight the potential effects on economic growth of changing energy resource abundance and scarcity, and other key parameters. We name and define them as follows:

b. Abundant Wood: Wood quantity is 10 times higher than in the Baseline scenario, so that $\bar{E}_M = 10$ instead of 1 in table 1's Baseline scenario.
c. Expensive Coal: The coal price is doubled, so that $\bar{e}_S = 0.23$ instead of 0.115.
d. More Substitutability: The elasticity of inter-sectoral substitution σ is increased to 4.2 instead of 4.
e. Less Substitutability: The elasticity of inter-sectoral substitution σ is reduced to 3.5 instead of 4.
f. Low Solow Knowledge: The initial stock of Solow sector varieties, $N_{S,0}$, is halved to 0.5 instead of 1.
g. Constant Population: $L_t = 1$ always, instead of $L_t \equiv S_t/S_1$ from (56).
h. Pre-industrial Stagnation: A combination of the Abundant Wood, More Substitutability, and Constant Population scenarios, one of many variants that results in a Pre-industrial Stagnation path.

Except in scenarios g. and h., we assume that population followed its historical path. In all cases apart from d., an Industrial Revolution is delayed, consistent with proposition 1: GDP per capita grows more slowly or declines, less labor shifts to the Solow sector, and coal use and growth are lower.

In the Abundant Wood scenario (fig. 8b) output per capita is nearly twice the Baseline level in 1560, and the wood price and (absolute) coal use are both much lower. Output per capita declines slowly as exogenous population grows, and while the price of wood rises steadily, by a factor of four from 1560 to 1900, at the end it does not reach even its initial Baseline level. Energy intensity (not shown) declines strongly, and the share of labor in the Malthus sector starts higher (95%) and falls much less, being still 73% in 1900. This scenario clearly illustrates the paradox where an abundance of wood stalls development despite much higher initial output per capita.

The Expensive Coal scenario (fig. 8c) is similar in some ways to the Abundant Wood scenario, but the initial income level is a little below the Baseline scenario and absolute coal use is even lower. The price of wood is about the same as coal, initially, so that here both fuels are relatively expensive, whereas in the Abundant Wood scenario, wood is much cheaper relative to output than in the Baseline scenario.

---

[32] We can fit some of the stylized facts better only at the expense of fitting others more poorly. For example, by reducing the initial number of varieties in the Malthus sector and therefore increasing the rate of growth in the Malthus sector we can obtain a more realistic trajectory for economic growth with more rapid early growth and slower later growth. However, because demand for wood remains stronger the price of wood does not peak in these scenarios. By contrast to obtain earlier more rapid growth in coal use we end up with even more growth concentrated into the 19th Century. Of course, if we allowed more parameters to vary freely rather than be set a priori, we could obtain a better fit to the data.

The More Substitutability scenario (fig. 8d) is an accelerated version of the Baseline Scenario. Output per capita increases 16-fold and coal use increases 750-fold overall. The former is far more than occurred in reality and the latter exactly matches the real-world increase. The wood price falls more steeply after its peak than in the Baseline. This might seem counter-intuitive, as a high elasticity of substitution can give rise to Pre-industrial Stagnation in our model, but it also increases the asymptotic rate of growth on an Industrial Revolution path, as predicted by de la Grandville (1989) and in our proposition 4.

By contrast, in the Less Substitutability Scenario (fig. 8e), growth is delayed, and the wood price rises more than in the Baseline. Assuming the economy had low Solow sector knowledge in 1560 produces similar results (fig. 8f) but with lower coal use.

The Constant Population and Pre-industrial Stagnation scenarios (figs. 8g-h) are very different to the other scenarios. Here there is extremely slow growth in GDP per capita and very little else changes. Constant Population starts from the same point as the Baseline scenario, while Pre-industrial Stagnation starts from a different point than any other scenario, consistent with proposition 2(a). Coal use is very slowly increasing under Constant Population, and very slowly declining under Pre-industrial Stagnation. We also ran the Pre-industrial Stagnation simulation with a low rate of population growth. Declining coal use is compatible with some amount of population growth. As the results look very similar to fig. 8h we have not included them here.

7. Discussion and Conclusions

We have shown the potential importance of the differential abundance of renewable and fossil energy resources − wood and coal − in driving the historic transition to modern economic growth, using a model that both yields analytical insights and reproduces key empirical features of the British Industrial Revolution. We extended and calibrated an increasing machine-varieties, directed technical change model, which, unlike previous related research, does not assume productivity or its growth to be inherently higher in the modern, industrial, coal-using, “Solow” sector than in the traditional, “wood”-using “Malthus” sector. Rather, we assume that resource supply conditions differ inherently, so that wood is inelastically and coal elastically supplied, which is a stylized representation of the British historical record.

Comparative static analysis of our model showed the effect of key parameters on the economy’s state of development: notably, any of a lower coal price, lower wood quantity, or higher population will further industrialize the economy. Our model’s dynamic analytical results show that growth is forever unbalanced, and there is an underlying tendency towards coal-biased growth, stemming from our asymmetric assumptions about the supply of wood and coal. If the elasticity of substitution between wood-intensive and coal-intensive goods is relatively low, then an industrial revolution, where production increasingly uses coal rather than wood, is inevitable. But if the elasticity is high enough and wood is initially sufficiently abundant relative to coal, then thanks to intersectoral labor mobility, the market size effect is strong enough so that the incentives for innovation more and more favor wood-directed innovation, despite diminishing direct innovation returns to accumulated knowledge, resulting in wood-biased technical change forever, which we call Pre-industrial Stagnation. There is then a separatrix between development paths that start with sufficiently abundant wood and those that do not.

If the asymptotic rate of economic growth in an Industrial Revolution path is constant, then the asymptotic rate under Pre-industrial Stagnation is zero. Pre-industrial Stagnation is also typically characterized by Strong Bias, defined as wood use relative to coal rising despite the relative price of wood rising. For "medium" and "high" levels of substitutability an Industrial Revolution eventually has Strong Bias too, with relative coal use and price both rising.

Given some parameter values from the literature, fitting our model to some basic stylized historical facts results in a baseline simulation with sensible values for the free parameters, and a development path that reproduces the key features of the British Industrial Revolution. From the start, the growth rate of coal-using machine varieties exceeds that of wood-using varieties, though its absolute growth is less until 1820. The only exogenous driver in our model is the historical rate of population growth. This should be endogenized in future research but leaving it exogenous here better highlights the role of natural resource scarcity in driving growth. The rate of economic growth accelerates partly because of the shift to the coal-using sector and partly because increased population increases the rate of innovation.

Compared to the previous literature (see Ashraf and Galor, 2011), our model introduces a new reason for why an economy may either remain forever in Pre-industrial Stagnation or fail to make a timely Industrial Revolution, since either may be caused by abundant wood, high elasticities of substitution and/or slow population growth. Our model's counterfactual simulations show that a much higher fixed quantity of wood input or fixed price of coal, and/or slower population growth would have greatly delayed growth of GDP per capita and the rate of innovation. In our model, it is the growing relative scarcity of wood caused by population growth that results in innovation to develop coal-using machines. Necessity is thus indeed the mother of invention: on its own, the unlimited supply of coal does not trigger a transition if wood is not relatively scarce.

Our model thus partly supports views by Allen (2009) and Wrigley (2010) that the Industrial Revolution first happened in Britain mainly because of its cheap, abundant coal. Counter to Clark and Jacks (2007), Madsen *et al*. (2010), and Harley and Crafts (2000), our model tells a plausible story of how coal could have played a central role in the Industrial Revolution.

However, we stress that our support is partial, because our model does not imply that cheap coal alone would have been sufficient for the Industrial Revolution to happen in Britain in the 18th Century. Improved institutions or growth in human capital (Clark, 2014), differences in demography (Voigtländer and Voth, 2006), and even the location of monasteries (Andersen *et al*., 2017) have all been suggested as key factors that might explain why the Industrial Revolution happened when and where it did. Our results should not be seen as disagreeing with these views. Institutional factors are invisibly assumed in the mathematical structure of most economic growth models, including ours, so we implicitly treat them as also being necessary for growth. If economic analysis can take these factors as well as renewable energy scarcity and fossil fuel availability all into account, then the Industrial Revolution may not "remain[s] one of history's mysteries" (Clark, 2014, p.260) for much longer.

References

Acemoglu, Daron. 2002. Directed technical change. *Review of Economic Studies* 69: 781–810.

Acemoglu, Daron. 2007. Equilibrium bias of technology. *Econometrica* 75(5): 1371–1409.

Acemoglu, Daron, Philippe Aghion, Leonardo Bursztyn, and David Hemous. 2012. The environment and directed technical change. *American Economic Review* 102(1): 131–66.

Allen, Robert C. 2009. *The British Industrial Revolution in global perspective*. Cambridge: Cambridge University Press.

Allen, Robert C. 2012. The shift to coal and implications for the next energy transition. *Energy Policy* 50: 17–23.

Andersen, Thomas Barnebeck, Jeanet Bentzen, Carl-Johan Dalgaard, and Paul Sharp. 2017. Pre-reformation roots of the Protestant Ethic. *Economic Journal* 127(604): 1756–93.

Ashraf, Quamrul, and Oded Galor. 2011. Dynamics and stagnation in the Malthusian epoch. *American Economic Review* 101: 2003–41.

Barbier, Edward B. 2011. *Scarcity and frontiers: How economies have developed through natural resource exploitation.* Cambridge and New York; Cambridge University Press.

Boserup, Ester. 1981. *Population and technological change: A study of long-term trends*. Chicago: Chicago University Press.

Broadberry, Stephen, Bruce M. S. Campbell, Alexander Klein, Mark Overton, and Bas van Leeuwen. 2015. *British economic growth*, *1270–1870*. Cambridge: Cambridge University Press. Database downloaded from https:/www.nuffield.ox.ac.uk/People/sites/stephen.broadberry/SitePages/Biography.aspx

Clark, Gregory. 2010. The macroeconomic aggregates for England, 1209-2008. *Research in Economic History* 27: 51–140.

Clark, Gregory. 2014. The Industrial Revolution. In *Handbook of economic growth,* Vol 2A, ed. Philippe Aghion and Steven Durlauf, 217–62. Amsterdam: North Holland.

Clark, Gregory, and David Jacks. 2007. Coal and the Industrial Revolution 1700-1869. *European Review of Economic History* 11: 39–72.

Crafts, Nicholas. 2010. Explaining the First Industrial Revolution: Two views. *European Review of Economic History* 15: 153–68.

de La Grandville, O., 1989. In quest of the Slutsky diamond. *American Economic Review* 79(3):.468-481.

Eren, Egemen, and Daniel Garcia-Macia. 2013. *From wood to coal may well be from Malthus to Solow*. Typescript, Department of Economics, Stanford University.

Fernihough, Alan, and Kevin Hjortshøj O'Rourke. 2014. Coal and the European Industrial Revolution. NBER Working Paper 19802.

Fouquet, Roger. 2011. Divergences in long run trends in the prices of energy and energy services. *Review of Environmental Economics and Policy* 5(2): 196–218.

Fouquet, Roger, and Stephen Broadberry. 2015. Seven centuries of European economic growth and decline. *Journal of Economic Perspectives* 29(4): 227–44.

Fröling, Maria. 2011. Energy use, population and growth, 1800-1970. *Journal of Population Economics* 24(3): 1133–63.

Gancia, Gino, and Fabrizio Zilibotti. 2005. Horizontal Innovation in the Theory of Growth and Development. In *Handbook of economic growth,* Vol 1A, ed. Philippe Aghion and Steven Durlauf, 111–70. Amsterdam: North Holland.

Gars, Johan, and Conny Olovsson. 2019. Fuel for economic growth? *Journal of Economic Theory* 184: 104941.

Gentvilaite, Ruta, Astrid Kander, and Paul Warde. 2015. The role of energy quality in shaping long-term energy intensity in Europe. *Energies* 8(1): 133–53.

Gollin, Douglas. 2014. The Lewis model: A 60-year retrospective. *Journal of Economic Perspective*s 28(3): 71–88.

Hanlon, W. Walker. 2015. Necessity is the mother of invention: Input supplies and directed technical change. *Econometrica* 83: 67–100.

Hansen, Gary D., and Edward C. Prescott. 2002. From Malthus to Solow. *American Economic Review* 92(4): 1205–17.

Harley, C. Knick, and Nicholas F. R. Crafts. 2000. Simulating the two views of the British Industrial Revolution. *Journal of Economic History* 60: 819–41.

Hart, R. 2013. Directed technological change and factor shares. *Economics Letters* 119(1): 77–80.

Hills, Sally, Ryland Thomas, and Nicholas Dimsdale. 2010. The UK recession in context—What do three centuries of data tell us? *Bank of England Quarterly Bulletin* 2010 Q4: 277–88.

Iriarte-Goñi, Iñaki, and María-Isabel Ayuda. 2012. Not only subterranean forests: Wood consumption and economic development in Britain (1850–1938). *Ecological Economics* 77: 176–84.

Kaldor, Nicholas (1963). Capital accumulation and economic growth. In *The theory of capital: Proceedings of a conference*, ed. F. A. Lutz and D. C. Hague, 177–222. Macmillan.

Kander, Astrid, Paolo Malanima, and Paul Warde. 2014. *Power to the people – Energy and economic transformation of Europe over four centuries*. Princeton, NJ: Princeton University Press.

Kander, Astrid, and David I. Stern. 2014. Economic growth and the transition from traditional to modern energy in Sweden. *Energy Economics* 46: 56–65.

Kelly, Morgan, Joel Mokyr, and Cormac Ó Gráda. 2015. Roots of the Industrial Revolution. UCD Centre for Economic Research Working Paper WP2015/24.

Khan, B. Zorina, and Kenneth L. Sokoloff. 2004. Institutions and technological innovation during early economic growth: Evidence from the great inventors of the United States, 1790-1930. NBER Working Paper 10966.

Klump, Rainer, Peter McAdam, and Alpo Willman. 2012. The normalized CES production function: theory and empirics. *Journal of Economic Surveys* 26(5): 769–99.

Kögel, Tomas, and Alexia Prskawetz. 2001. Agricultural productivity growth and escape from the Malthusian Trap. *Journal of Economic Growth* 6: 337–57.

Krausmann, Fridolin, Heinz Schandl, and Rolf Peter Sieferle. 2008. Socio-ecological regime transitions in Austria and the United Kingdom. *Ecological Economics* 65: 187–201.

Kunnas, Jan, and Timo Myllyntaus. 2009. Postponed leap in carbon dioxide emissions: The impact of energy efficiency, fuel choices and industrial structure on the Finnish economy 1800-2005. *Global Environment* 3: 154–89.

Lemoine, Derek. 2017. Innovation-led transitions in energy supply. NBER Working Paper 23420.

Lewis, W. Arthur. 1954. Economic development with unlimited supplies of labor. *Manchester School* 22(2): 139–91.

Madsen, Jakob B., James B. Ang, and Rajabrata Banerjee. 2010. Four centuries of British economic growth: The roles of technology and population. *Journal of Economic Growth* 15(4): 263–90.

Marchetti, Cesare, Perrin S. Meyer, and Jesse H. Ausubel. 1996. Human population dynamics revisited with the logistic model: How much can be modeled and predicted? *Technological Forecasting and Social Change* 52: 1–30.

Mokyr, Joel. 2009a. Intellectual property rights, the Industrial Revolution, and the beginnings of modern economic growth. *American Economic Review* 99(2): 349–55.

Mokyr, Joel. 2009b. *The enlightened economy: An economic history of Britain 1700–1850*. New Haven: Yale University Press.

O'Rourke, Kevin Hjortshøj, Ahmed S. Rahman, and Alan M. Taylor. 2013. Luddites, the Industrial Revolution, and the demographic transition. *Journal of Economic Growth* 18: 373–409.

Pomeranz, Kenneth L. 2001. *The great divergence: China, Europe and the making of the modern world economy.* Princeton, NJ: Princeton University Press.

Stern, David I., and Astrid Kander. 2012. The role of energy in the Industrial Revolution and modern economic growth. *Energy Journal* 33(3): 125–52.

Strulik, Holger, and Jacob Weisdorf. 2008. Population, food, and knowledge: A simple unified growth theory. *Journal of Economic Growth* 13: 105–216.

Tahvonen, Olli, and Seppo Salo. 2001. Economic growth and transitions between renewable and nonrenewable energy resources. *European Economic Review* 45: 1379–98.

Thomas, Brinley. 1986. Was there an energy crisis in Great Britain in the 17th Century? *Explorations in Economic History* 23: 124–52.

Voigtländer, Nico, and Hans-Joachim Voth. 2006. Why England? demographic factors, structural change and physical capital accumulation during The Industrial Revolution. *Journal of Economic Growth* 11(4): 319–61.

Warde, Paul. 2007. *Energy consumption in England and Wales 1560-2000*. Napoli: ISSM-CNR.

Wilkinson, Richard G. 1973. *Poverty and progress: An ecological model of economic development*. London: Methuen.

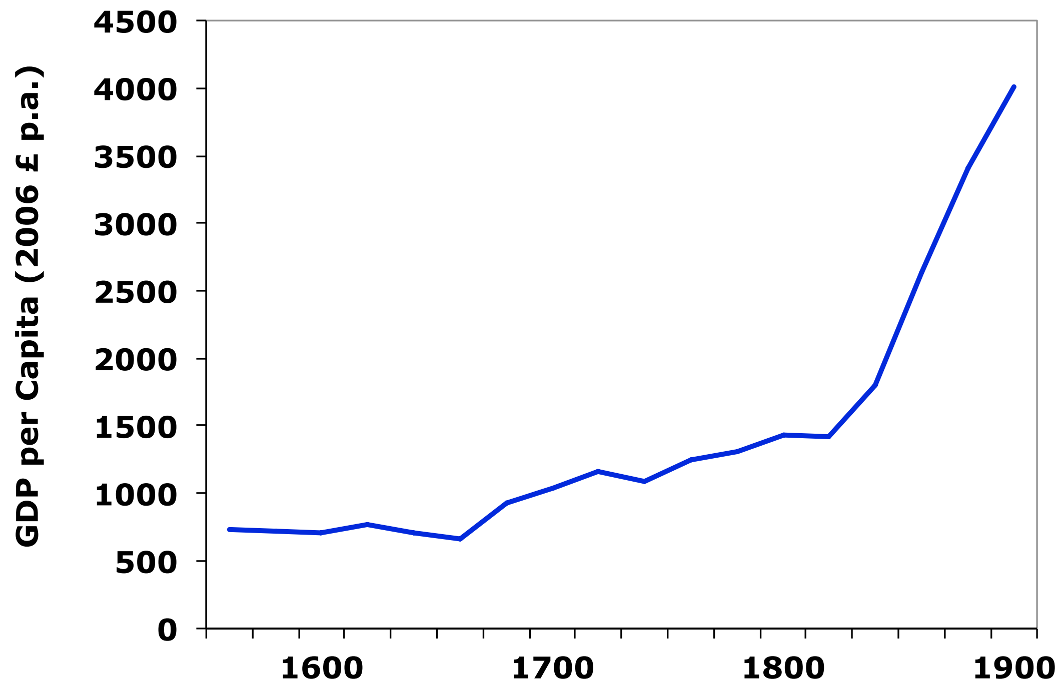
Wrigley, E. Anthony. 1988. *Continuity, chance, and change: The character of the Industrial Revolution in England*. Cambridge: Cambridge University Press.

Wrigley, E. Anthony. 2010. *Energy and the English Industrial Revolution*. Cambridge: Cambridge University Press.

Table 1. Baseline Parameters

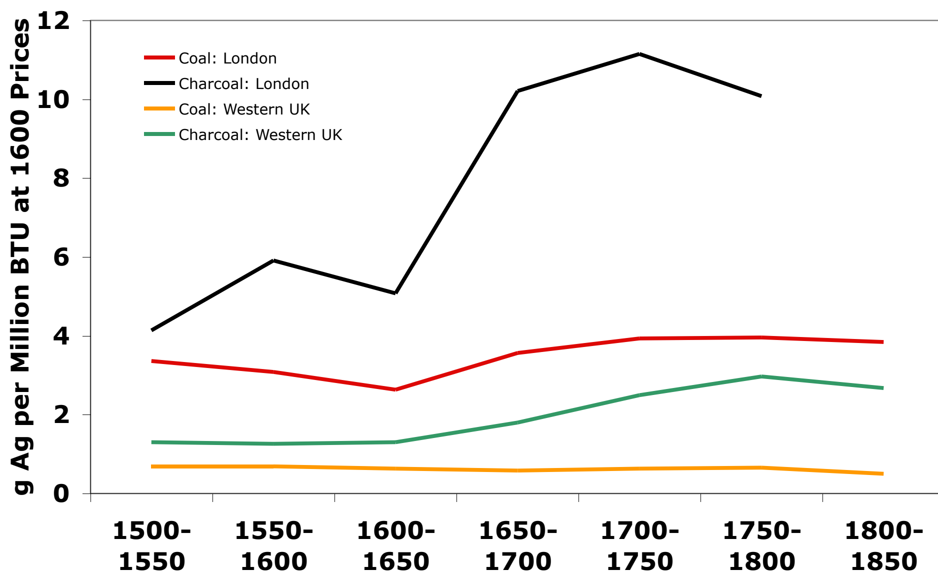
Parameter	**Symbol**	**Value**	**Sources**
CES elasticity in final production	σ	4	Greater than $\sigma^{\dagger}$
Distribution parameter in CES final production	γ	0.853	Optimized
Energy output elasticity	α	0.25	Energy cost share in 1800 in the UK was about 25% not counting animal and human power (Gentvilaite et al., 2015).
Capital (machine) output elasticity	β	0.225	This is based on a share of capital that fluctuates between about 0.2 and 0.25 in Clark (2010).
Productivity innovation in M sector	η_M	2.26	Optimized
Productivity innovation in S sector	η_S	2.26	Optimized
Initial idea stock in M sector	$N_{M,0}$	5	Optimized
Initial idea stock in S sector	$N_{S,0}$	1	Normalized
Constant price of coal	$\bar{e}_S$	0.115	Optimized
Constant consumption of wood	$\bar{E}_M$	1	Normalized

Figure 1. GDP per Capita



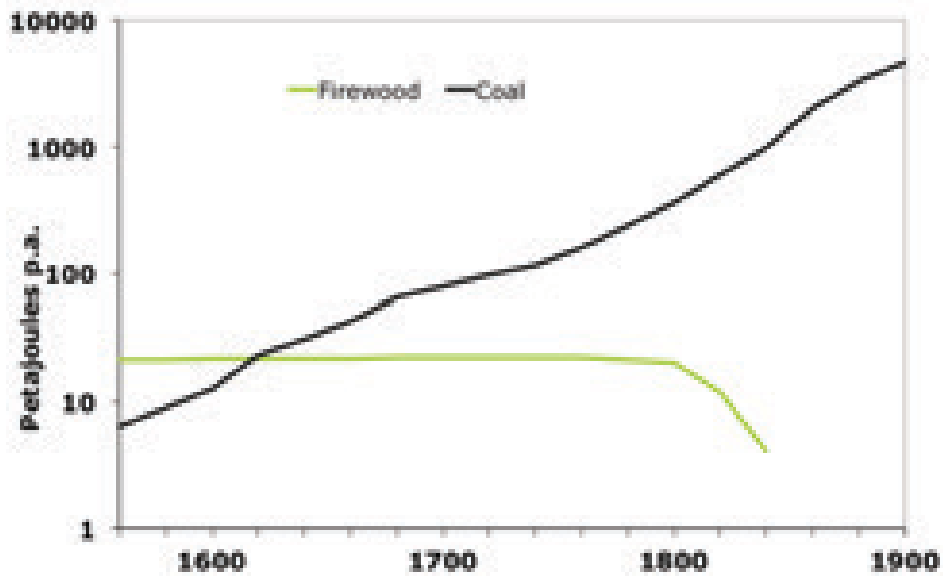
Source: Broadberry *et al*. (2015).

Figure 2. Real Prices of Coal and Charcoal in London and the Western UK



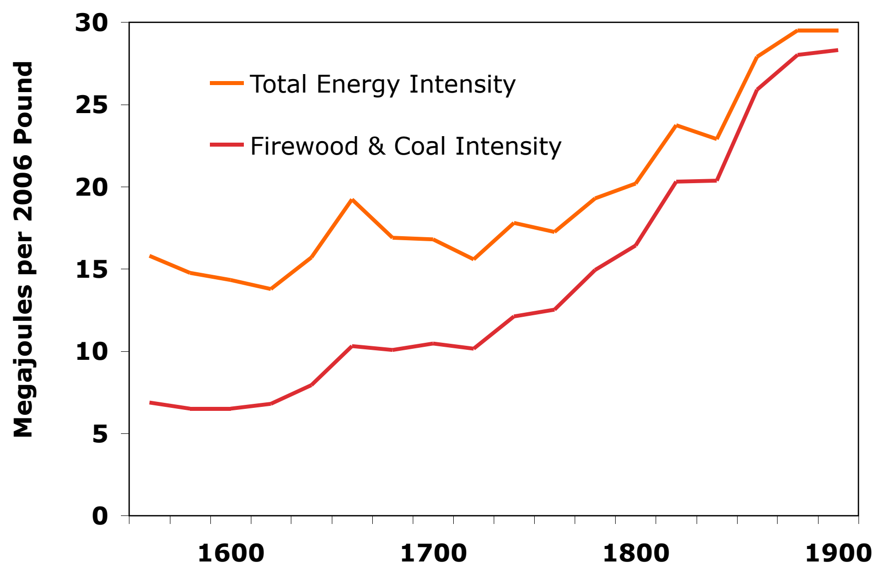
Source: Allen (2009), Table 4.3. Units are grams of silver per million BTU at constant prices of 1550.

Figure 3. Quantities of Firewood and Coal



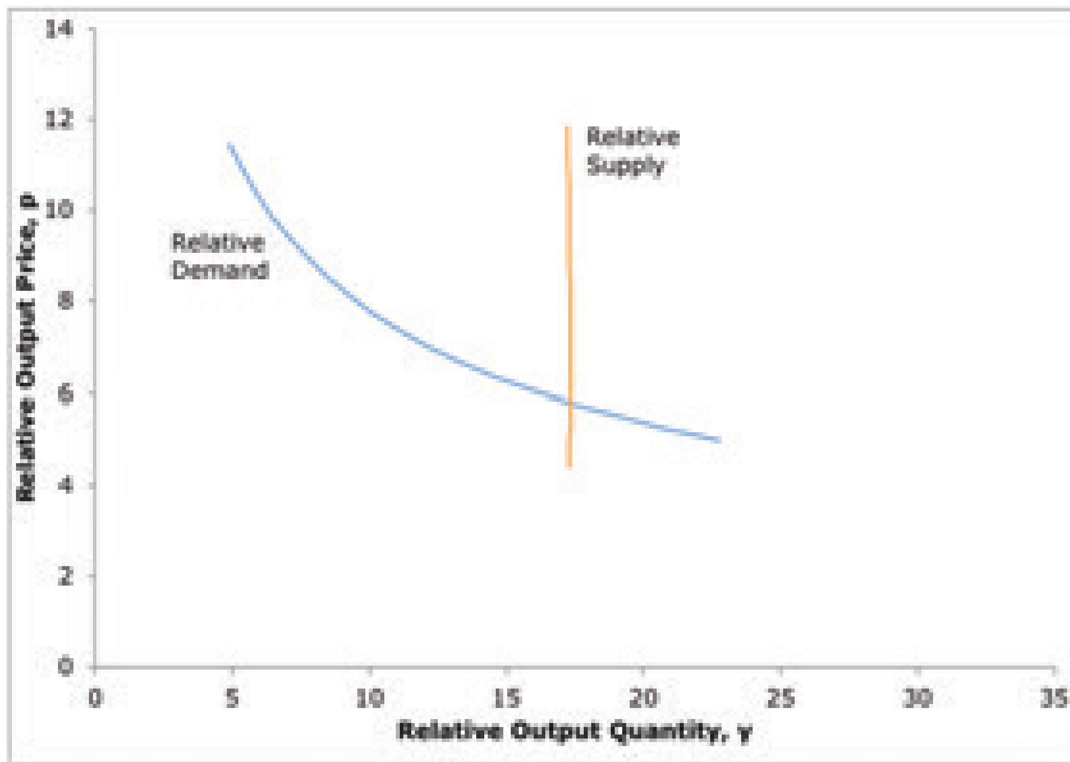
Source: Warde (2007).

Figure 4. Energy Intensity

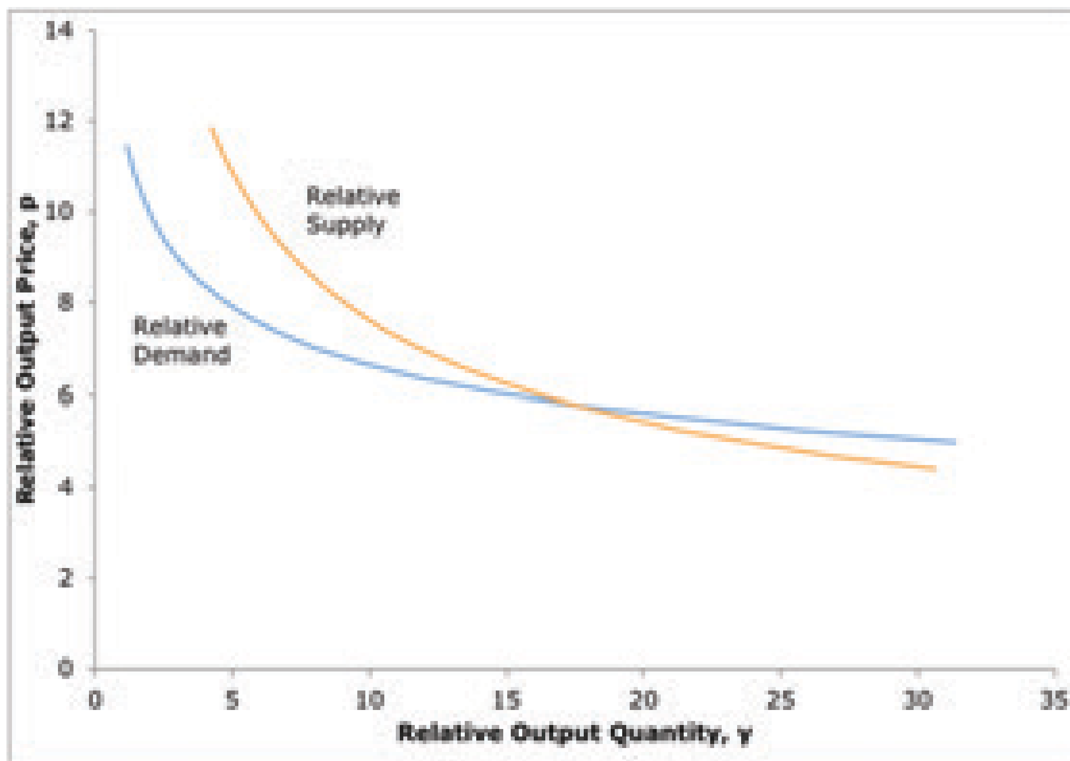


Sources: Authors' calculations from data in Warde (2007), Broadberry *et al*. (2015), and Hills *et al*. (2010).

Figure 5. Relative Output Supply and Demand



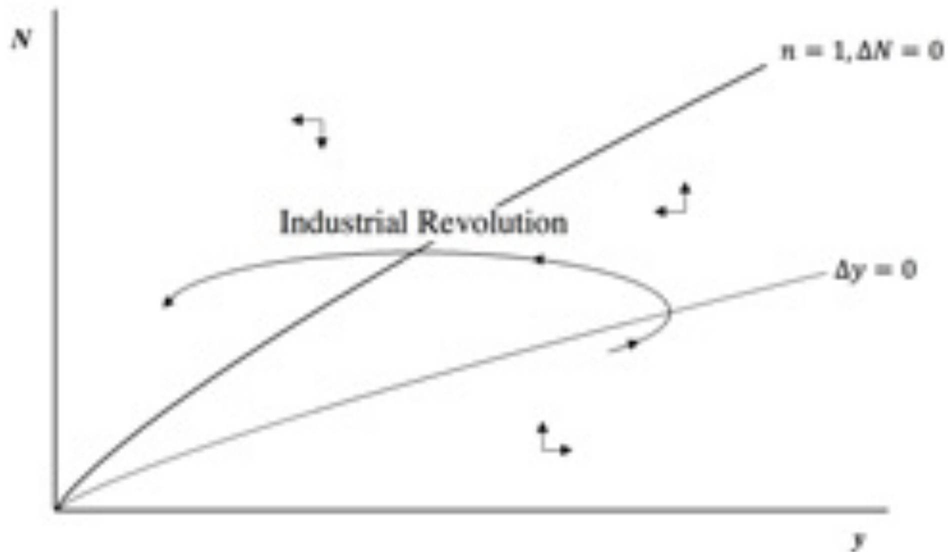
a. $\sigma = 1.85$



b. $\sigma = 4$

Figure 6. Phase Diagrams in Malthus/Solow Machine Varieties ratio, *N*, and Goods Ratio, *y*, Space

a. Medium Elasticity of Substitution ($\tilde{\sigma} < \sigma < \sigma^{\dagger}$)



b. High Elasticity of Substitution ($\sigma > \sigma^{\dagger}$)

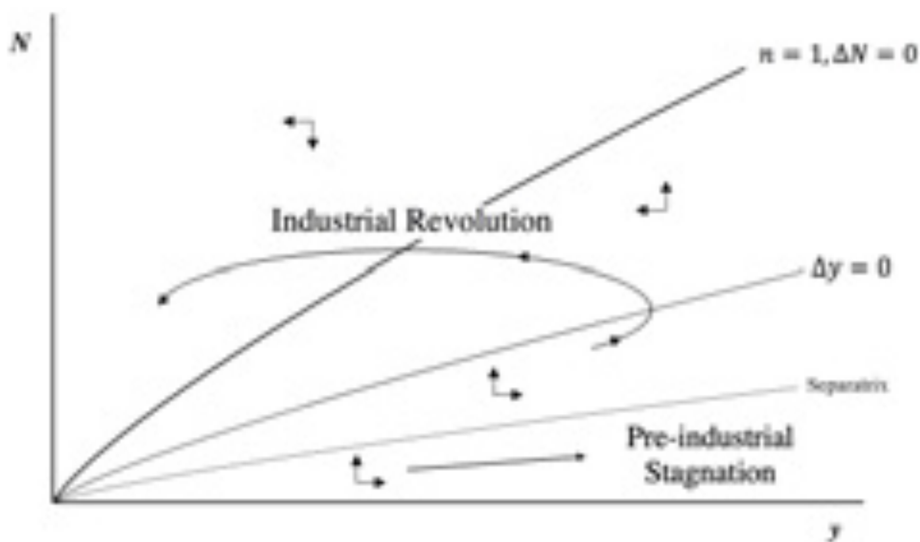
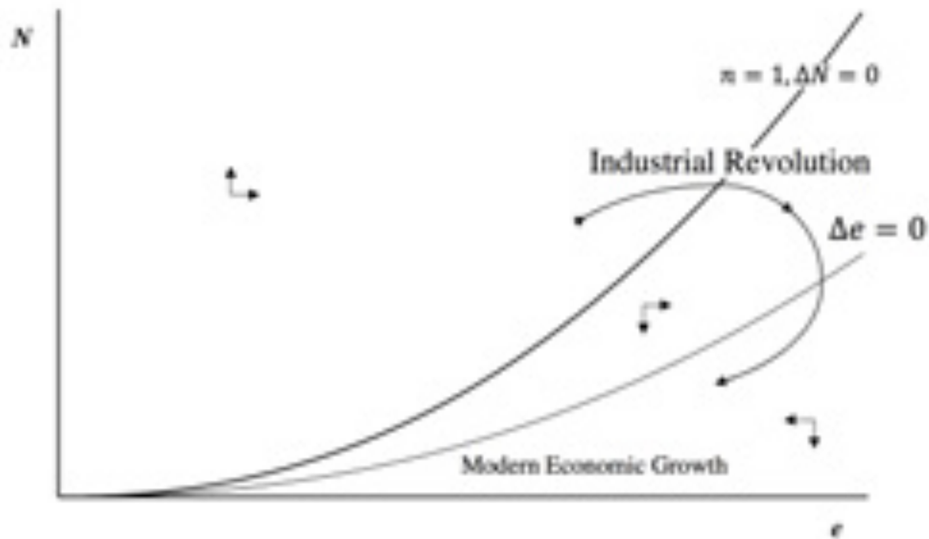


Figure 7. Phase Diagrams in Malthus/Solow Machine Varieties Ratio, *N*, and Energy Price Ratio, *e*, Space

a. Medium Elasticity of Substitution ($\tilde{\sigma} < \sigma < \sigma^{\dagger}$)



b. High Elasticity of Substitution ($\sigma > \sigma^{\dagger}$)

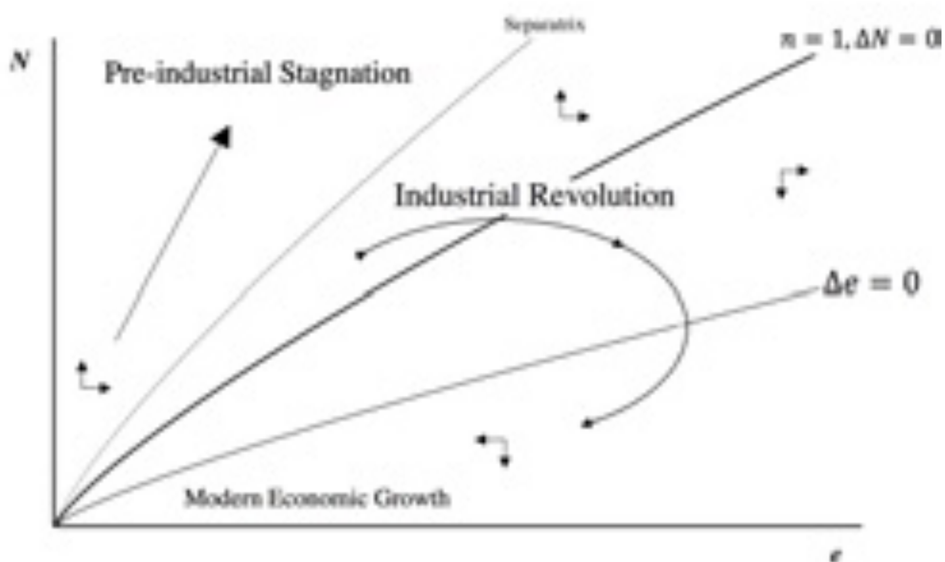


Figure 8. Baseline and Counterfactual Simulations