A Continuous-Time Asset Pricing Model with Smooth Ambiguity Preferences*

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Abstract

This study extends the smooth ambiguity preferences model proposed by Klibanoff et al. (2005, 2009) to a continuous-time dynamic setting. It is known that these preferences converge to the subjective expected utility as the time interval shortens so that decision makers do not exhibit any ambiguity-sensitive behavior in the continuous-time limit. Accordingly, this study proposes an alternative model of decision making that applies Yaari's (1987) dual theory to the original preferences and interchanges the role of the second-order utility function with that of the second-order probability. We then formulate a recursive utility of smooth ambiguity-sensitive decision makers in continuous-time. Our model is represented by the stochastic differential utility with distorted beliefs so that most existing techniques in financial studies can be made applicable together with these distorted beliefs. We give an asset-pricing example to demonstrate the applicability of our model.

Keywords: Smooth Ambiguity Preferences · Continuous Time · Asset Pricing · Recursive Utility

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1 Introduction

For a long time, the subjective expected utility (SEU) has been the standard theory of decision-making under uncertainty. According to this theory, even when a decision maker (DM) does not know the objective probability for uncertain payoffs, he/she makes a decision based on a unique subjective probability measure. Therefore, the DM should be indifferent between a "risky" bet where the objective probability measure is known and an "ambiguous" bet where that measure is unknown.

However, Ellsberg (1961) challenges the SEU as a normative theory of decision-making. Based on a well-designed thought experiment, he argues that the DM would be ambiguity-averse in most cases; that is, most DMs would prefer a risky bet over an ambiguous bet. Subsequent experimental studies have supported Ellsberg's conjecture, and many theoretical models incorporating DM's attitude toward ambiguity as well as risk have been developed.

Among these theories, Klibanoff et al. (2005) develop a so-called smooth ambiguity preferences model.[1] Assuming multiple probabilities for relevant payoffs, they model the DM's uncertainty attitude through double expected utilities. In the KMM model, the DM subjectively estimates the relative likelihood of each probability measure being the true measure. These likelihoods are called the second-order probability, whereas each probability measure for the relevant payoffs is called the first-order probability. In the first stage of the KMM model, the DM calculates his/her expectations of the first utility function of the relevant payoffs under each first-order probability. In the second stage, the DM employs the second-order probability to calculate his/her expectation of the second utility function of the first-stage expected utilities. Klibanoff et al. (2009) extended their KMM model to a dynamic model under a discrete-time setting.

However, Skiadas (2013) has shown that the KMM preferences converge to the SEU in the continuous-time limit. An implication for Skiadas's result is as follows: as the Arrow-Pratt risk premium depends on the variance of risky payoffs, the ambiguity premium implied by the KMM preferences depends on the variance of the first stage expected utilities. Because the latter variance is proportional to the square of the time interval under both the Brownian and Poisson uncertainties, the ambiguity premium becomes negligible when we consider the continuous-time limit.

[1] Other notable ambiguity-sensitive preferences include the maxmin utility of Gilboa and Schmeidler (1989), the Choquet expected utility of Schmeidler (1989), the multiplier preferences of Hansen and Sargent (2001) and Hansen et al. (1999), and the variational preferences of Maccheroni et al. (2006).

Given that many powerful techniques are aligned with continuous-time financial modeling, this study combines those with the KMM preferences. For this purpose, we apply Yaari's (1987) "dual theory" to the KMM model and interchange the role of the second utility in the original KMM preferences with that of the second-order probability. Using this trick, we prevent the DM's ambiguity attitude from disappearing in the continuous-time limit. Then, by using the stochastic differential utility (SDU), we achieve a three-way separation of the DM's preferences: attitude toward risk, intertemporal substitution, and ambiguity. While this study concentrates on the Brownian uncertainty case, our model could also incorporate Poisson jump processes.

Gindrat and Lefoll (2011) also extend the KMM preferences to the continuous-time dynamic setting. They achieve this extension by assuming that the DM's ambiguity aversion coefficient becomes infinity as the time interval shortens to zero. In contrast, the degree of the DM's ambiguity aversion in our model can be well defined independently from the time interval. We believe that this difference is critical because attitude toward ambiguity as well as risk should be treated as the DM's intrinsic nature. Therefore, a model that enables us to conduct a frequency-independent comparative analysis of the DM's ambiguity attitude would be more reasonable.

The DM's preferences in our model are eventually represented by the SEU with distorted beliefs.[2] In particular, our ambiguity-averse DM behaves as an SEU agent with pessimistic beliefs. In this sense, our study could possibly be classified according to the distorted beliefs models by Abel (2002) and Cecchetti et al. (2000). However, the distorted beliefs in our model result from the DM's rational response to ambiguity, whereas those beliefs are completely exogenous in the other distorted beliefs models.

We apply our model to asset pricing and calibrate our model under a setting that is the continuous-time analogue of Ju and Miao's (2012) model. Under the discrete-time setting, Ju and Miao (2012) indicate that the KMM preferences help resolve many asset pricing puzzles. Our calibration results reveal that our model takes over these properties. Furthermore, owing to the continuous-time modeling, we can explicitly decompose the equity premium into three parts: the risk premium, the premium for late resolution of uncertainty, and the ambiguity premium.

The remainder of this study is constructed as follows. Section 2 introduces the recursive utility and the

[2] Gollier (2011) indicates that, at the optimum, the DM's behavior under the KMM model is observationally equivalent to that under the SEU model with distorted beliefs.

KMM smooth ambiguity preferences under a discrete-time setting. Section 3 derives the continuous-time smooth ambiguity preferences and formalize the DM's ambiguity aversion in our model. Section 4 applies our model to asset pricing and indicates that it generates very similar results to those of Ju and Miao (2012). Section 5 concludes the study.

2 Discrete-Time Smooth Ambiguity Preferences

We first consider the DM's preferences under a discrete-time setting to formulate our continuous-time model. Time is indexed by $t = 0, h, 2h, \cdots, Nh$, where $h \in (0,1)$ represents the time interval. The state space and filtration are denoted by $(\Omega, \{\mathscr{F}_t\}_{t=0}^{T})$, where T is the terminal date; that is, $T = Nh$.

2.1 Recursive Utility without Ambiguity

The DM ranks an adapted consumption process, $C_t \in (0,\infty)$, and his/her time-t utility of the continuation consumption stream, $\{C_s\}_{s=t}^{T}$, is denoted by V_t. In particular, this study considers the following recursive utility proposed by Epstein and Zin (1989):

$$V_t = W\left(C_t, m_t\left(V_{t+h}\right)\right). \tag{1}$$

In equation (1), $m_t\left(V_{t+h}\right)$ denotes a time-t certainty equivalent (CE) for the uncertain next period utility, V_{t+h}. It is assumed that $V_t(\omega) \in (0,\infty)$ for $\forall (t,\omega)$.

Supposedly, Kreps and Porteus's (1978) form of CE would be the most commonly used in economics and finance studies. Under the setting without ambiguity, this CE is described as follows:

$$m_t^{\mathrm{KP}}\left(V_{t+h}\right) = u^{-1}\left(E_t\left[u\left(V_{t+h}\right)\right]\right),$$

where $E_t[\cdot] \equiv E[\cdot \,|\, \mathscr{F}_t]$ denotes the time-t conditional expectation under a probability measure, $\mathbb{P}$, and $u(\cdot)$ denotes a von Nuemann-Morgenstern utility index that determines the DM's risk attitude. We assume that $u(\cdot)$ is a strictly increasing, continuous, and three times continuously differentiable function. Meanwhile, $W(\cdot,\cdot)$ is called the aggregator that represents the DM's intertemporal attitude under the deterministic setting. Therefore, the recursive utility in equation (1) allows us to separate the DM's intertemporal attitude from his/her risk attitude.

2.2 Original KMM Model

Next, we introduce ambiguity, which is the key feature of this study. With ambiguity, the DM cannot identify the true probability measure on $(\Omega, \mathscr{F}_T)$. Instead, the DM assumes multiple first-order probability measures on $(\Omega, \mathscr{F}_T)$. We define Δ as the set of the first-order probability measures, $\{\mathbb{Q}^\theta\}_{\theta\in\Theta}$, where Θ is a parameter space. It is assumed that all $\mathbb{Q}^\theta \in \Delta$ are mutually equivalent. For notational convenience, one of the first-order probabilities is selected as the DM's reference probability measure and is denoted by $\mathbb{P}$.[3] The conditional expectation under $\mathbb{P}$ is denoted by $E_t[\cdot]$, whereas that under $\mathbb{Q}^\theta$ is denoted by $E_t^\theta[\cdot]$. The Radon-Nikodým derivative process of $\mathbb{Q}^\theta$ with respect to $\mathbb{P}$, denoted by $\xi_t^\theta \equiv (d\mathbb{Q}^\theta/d\mathbb{P})\big|_{\mathscr{F}_t}$, depends on the corresponding adapted parameter process, $\theta = \{\theta_t\}_{t=0}^T$, which generates the time-t conditional expectation of any $\mathscr{F}_u$-measurable random variable X_u (where $u > t$) under $\mathbb{Q}^\theta$ through $E_t^\theta[X_u] = E_t\left[\xi_u^\theta X_u\right]/\xi_t^\theta$.

Because the DM is uncertain about which probability measure, $\mathbb{Q}^\theta \in \Delta$, governs the real world, he/she attaches at each t the subjective probability weights, $\eta_t(\theta) \geq 0$, to each $\theta \in \Theta$, where $\int_\Theta \eta_t(\theta)\, d\theta = 1$.[4] We refer to η_t as the DM's second-order probability. Given η_0, the second-order probability, η_t, is the time-t Bayesian update of η_0; that is,

$$\eta_t(\theta') = \frac{\xi_t^{\theta'}\eta_0(\theta')}{\int_\Theta \xi_t^\theta \eta_0(\theta)\, d\theta}, \quad \text{for } \forall\theta' \in \Theta.$$

From this update rule of η_t, the compound probability measure, $\mathbb{Q}^{\overline{\theta}}(\omega)\Big|_{\mathscr{F}_t} \equiv \int_\Theta \mathbb{Q}^\theta(\omega)\,\eta_t(\theta)\, d\theta$ for $\forall\omega \in \Omega$, could be well defined on $(\Omega, \mathscr{F}_T)$.

Figure 1 illustrates the time period between t and $t+h$ in the above setting. Suppose that the present time is t and the DM's current consumption and utility are C_t and V_t, respectively, and both are $\mathscr{F}_t$-measurable. For the moment, the number of states at time $t+h$ that are accessible from the current state is assumed to be finite and those states are denoted by $s_{t+h|t} = 1, 2, \cdots, J$. We also temporarily assume that the parameter

[3]The choice of the reference probability measure is arbitrary, and $\mathbb{P}$ does not necessarily represent the true probability measure.
[4]When Θ is a finite space, $\eta_t(\cdot)$ can be interpreted as a probability mass function that satisfies $\sum_\Theta \eta_t(\theta) = 1$.

space, Θ, is finite, and the parameters are indexed by $\theta = 1, 2, \cdots, I$. In this illustration, the DM's next-period consumption and utility, $(C_{t+h}, V_{t+h}) \in \left\{ \left(c^1, v^1\right), \left(c^2, v^2\right), \cdots, \left(c^J, v^J\right) \right\}$, depend on $s_{t+h|t}$. In addition, there are multiple first-order probabilities, $\mathbb{Q}^{\theta}, \theta = 1, 2, \cdots, I$, and the DM is uncertain about which probability is the true measure for $s_{t+h|t}$. The DM then subjectively attaches the second-order probabilities, $\eta_t(\theta)$, to their respective first-order probabilities, $\mathbb{Q}^{\theta}$.

At time t, there are multiple CEs for V_{t+h} that are conditional on which first-order probability, $\mathbb{Q}^{\theta}$, is used for calculating the time-t conditional expectation. If the DM does not care about the variability in these conditional CEs, he/she will average these CEs using the second-order probability, η_t. This is equivalent to calculating the time-t CE for V_{t+h} directly through the compound probability, $\mathbb{Q}^{\overline{\theta}}\left(s_{t+h|t} \middle| \mathscr{F}_t \right) = \sum_{\theta=1}^{I} \mathbb{Q}^{\theta}\left(s_{t+h|t} \middle| \mathscr{F}_t \right) \eta_t(\theta)$. In this case, the DM is an SEU agent and referred to as ambiguity neutral. Meanwhile, if the DM is a maxmin utility agent, he/she will attach $\eta_t(\theta') = 1$ to the $\mathbb{Q}^{\theta'} \in \Delta$ that yields the lowest conditional CE.

Klibanoff et al. (2005, 2009) axiomatize the smooth ambiguity preferences that could be considered as an intermediate model between the two extremes, the SEU and the maxmin utility. In their model, the DM's time-t CE has the following form:

$$m_t^{\text{KMM}}(V_{t+h}) = v^{-1}\left(\int_{\Theta} v\left(m_t^{\theta}(V_{t+h}) \right) \eta_t(\theta)\, d\theta \right), \tag{2}$$

where $v(\cdot)$ is a strictly increasing, continuous, and at least twice continuously differentiable function, and

$$m_t^{\theta}(V_{t+h}) = u^{-1}\left(E_t^{\theta}\left[u(V_{t+h}) \right] \right), \tag{3}$$

represents the Kreps and Porteus (KP) CE conditional on $\mathbb{Q}^{\theta} \in \Delta$.

In the KMM CE, the function $v(\cdot)$, called the second-order utility, captures the DM's ambiguity attitude. It can be seen from equation (2) that the DM dislikes the variability in $m_t^{\theta}(V_{t+h})$ when $v(\cdot)$ is concave. Meanwhile, if $v(\cdot)$ is a linear function, the KMM CE is equivalent to the KP CE calculated under $\mathbb{Q}^{\overline{\theta}}$. In this case, the DM's utility is reduced to the SEU. The KMM preferences enable us to separate the degree of ambiguity measured by the variability in $m_t^{\theta}(V_{t+h})$ from the degree of the DM's ambiguity aversion measured by the concavity of $v(\cdot)$. Therefore, a comparative analysis of the DM's ambiguity aversion under the fixed degree of ambiguity can be conducted, which would be difficult in terms of the maxmin utility.

The KMM preferences are very flexible and tractable for representing the DM's ambiguity attitude. However, Skiadas (2013) reveals that the KMM CE degenerates into the KP CE as the time interval, h, shortens to zero.

Proposition 1 (Skiadas 2013)**.** *Under the discrete-time approximation of Brownian uncertainty, the KMM CE in equation (2) is expressed as follows:*

$$m_t^{\mathrm{KMM}}(V_{t+h}) = m_t^{\overline{\theta}}(V_{t+h}) + o(h),$$

where $m_t^{\overline{\theta}}(\cdot)$ is the time-t KP CE under the compound probability measure, $\mathbb{Q}^{\overline{\theta}}$.

Proposition 1 indicates that the role of the second-order utility, $v(\cdot)$, vanishes as h goes to zero. Under Brownian uncertainty, ambiguity is represented by multiple drifts of primitive processes; hence, a DM having the KMM smooth preferences should not care about the negligible variance of these drifts in the continuous-time limit.[5]

2.3 Dual Theory of a Dynamic KMM Model

The previous subsection indicates that the KMM preferences fail to capture the DM's ambiguity attitude in the continuous-time limit. We then propose an alternative dynamic smooth ambiguity preferences model that interchanges the role of the second-order utility, $v(\cdot)$, which is the key feature of the KMM model, with that of the second-order probability, η_t.

Under an atemporal setting without ambiguity, Yaari (1987) axiomatizes an alternative, so-called, dual theory of the SEU. Under the SEU model, the DM's risk attitude is represented by the concavity of the utility function $u(\cdot)$. In the dual theory, on the other hand, the DM's utility function is linear, but he/she evaluates its expected value using a distorted probability. Therefore, the DM's risk-attitude is captured through probability instead of a utility function in the dual theory. In particular, a risk-averse DM in the dual

[5] Skiadas (2013) also shows that the KMM CE degenerates into the KP CE under the Poisson uncertainty.

theory behaves like a pessimist in the sense that he/she distorts his/her subjective probabilities toward lower payoffs.

Applying Yaari's dual theory to an atemporal setting with ambiguity, Iwaki and Osaki (2014) developed the dual theory of KMM preferences. In their dual theory, the DM first calculates the conditional KP CE under each first-order probability by $m^{\theta}(C) = u^{-1}\left(E^{\theta}[u(C)]\right)$. The DM's second-order utility, $v(\cdot)$, is linear in $m^{\theta}(C)$, but he/she evaluates the expected value of $m^{\theta}(C)$ using a distorted second-order probability, denoted by $\tilde{\eta}$. The difference between η and $\tilde{\eta}$ captures DM's ambiguity attitude.

In the following, we apply Iwaki and Osaki's (2014) atemporal model to our dynamic setting. First, from the DM's second-order probability, η_t, we can define the decumulative distribution function of the time-t conditional KP CE, denoted by $G_t(\cdot)$, as follows:

$$G_t(z) \equiv \int_{\Theta} \mathbb{I}\left\{m_t^{\theta}(V_{t+h}) > z\right\} \eta_t(\theta)\, d\theta, \quad \text{for } z \in (0,\infty),$$

where $\mathbb{I}(\cdot)$ is an indicator function. We also define a transformation function $\varphi(\cdot) : [0,1] \to [0,1]$ that is continuous and strictly increasing. Through the transformation function, our ambiguity-sensitive DM distorts the decumulative distribution function of the time-t conditional KP CE by $\varphi[G_t(z)]$. We then propose the following time-t CE in our dynamic setting:

$$\begin{aligned} m_t^{\mathrm{D}}(V_{t+h}) &= \int_0^{\infty} \varphi[G_t(z)]\, dz \\ &= -\int_0^{\infty} z\, d\varphi[G_t(z)] \\ &= \int_{\Theta} m_t^{\theta}(V_{t+h})\, \tilde{\eta}_t(\theta)\, d\theta, \end{aligned} \tag{4}$$

where the second equality is obtained from integration by parts, and $\tilde{\eta}_t$ in the third line is the distorted second-order probability implied by $\varphi[G_t(z)]$.

In equation (4), the DM's time-t CE under the dual theory can be expressed by the expected value of the conditional KP CEs, $m_t^{\theta}(V_{t+h})$, where the expectation is calculated by $\tilde{\eta}_t$, instead of η_t. It can be seen from equation (4) that $m_t^{\mathrm{D}}(V_{t+h}) < m_t^{\overline{\theta}}(V_{t+h})$ holds if $\varphi(\cdot)$ is a convex function. In this case, $\tilde{\eta}_t$ attaches larger probability weights than η_t to $\mathbb{Q}^{\theta} \in \Delta$ that yield lower conditional KP CEs. Therefore, the DM dislikes any mean-preserving spread in the conditional KP CEs so that he/she is ambiguity averse.[6] From this response, the convexity of $\varphi(\cdot)$ represents the DM's ambiguity attitude in the dual theory.

The CE in equation (4) preserves the important property of the KMM CE. Through the dual theory, we can still separate the degree of ambiguity measured by the variability in $m_t^{\theta}(V_{t+h})$ from the degree of the DM's ambiguity aversion measured by the convexity of $\varphi(\cdot)$. Further, the dual theory as well as the KMM model need not assume an extremely ambiguity-averse DM as the maxmin utility. The advantage of the dual theory over the original KMM model is that $\tilde{\eta}_t$ does not degenerate into η_t even as h goes to zero because the distortion depends on the time-t distribution of θ_t. In the next section, we derive the utility process implied by the CE in equation (4) and show how well the dual theory can manage the DM's smooth ambiguity preferences under a continuous-time setting.

3 A Continuous-Time Smooth Ambiguity Preferences Model

This section develops a continuous-time smooth ambiguity preferences model. We consider a standard k-dimensional Brownian Motion (BM), $B_t = \left(B_t^1, B_t^2, \cdots, B_t^k\right)^{\top}$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathbb{P}$ is a reference probability measure. The filtration $\{\mathscr{F}_t\}_{t=0}^{T}$ is generated by B_t and $\mathbb{P}$-null sets of $\mathscr{F}$, and we assume $\mathscr{F}_T = \mathscr{F}$. As in the previous section, the DM ranks an adapted consumption process, $C_t \in (0, \infty)$, and his/her time-t conditional utility of continuation consumption streams, $\{C_s\}_{s=t}^{T}$, is denoted by V_t. The terminal value of utility is assumed to be zero; that is, $V_T = 0$.

3.1 Continuous-Time Ambiguity

We first formalize ambiguity under the continuous-time setting. For this, we define R^k-valued adapted processes, $\theta = \{\theta_t\}_{t=0}^{T}$, whose set is denoted by Θ. It is assumed that each θ satisfies the Novikov condition.

$$E_0\left[\exp\left\{\frac{1}{2}\int_0^T |\theta_s|^2 \, ds\right\}\right] < \infty.$$

[6] We define the DM's ambiguity aversion more precisely in the next section.

For each θ, the Radon-Nikodým derivative process is defined as follows:

$$\xi_t^\theta = \exp\left\{-\frac{1}{2}\int_0^t |\theta_s|^2 ds + \int_0^t \theta_s^\top dB_s\right\}, \quad \xi_0^\theta = 1.$$

We can then construct the first-order probability measures, $\mathbb{Q}^\theta$, corresponding to each θ by

$$\left.\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \xi_t^\theta, \quad \text{for } \theta \in \Theta.$$

Then, $B_t^\theta \equiv B_t - \int_0^t \theta_s ds$ is a standard BM under $\left(\Omega, \mathscr{F}, \mathbb{Q}^\theta\right)$.

By defining $\overline{\theta}_t \equiv \int_\Theta \theta_t \eta_t(\theta)\, d\theta$ as the time-t conditional expectation of θ_t under the second-order probability, we can construct the compound probability measure, $\mathbb{Q}^{\overline{\theta}}$, on $(\Omega, \mathscr{F})$, through the Radon-Nikodým derivative process, $\xi_t^{\overline{\theta}}$. Then, $B_t^{\overline{\theta}} \equiv B_t - \int_0^t \overline{\theta}_s ds$ is a standard BM under $(\Omega, \mathscr{F}, \mathbb{Q}^{\overline{\theta}})$.

Similarly, we can construct the distorted compound probability measure $\mathbb{Q}^{\tilde{\theta}}$ through the Radon-Nikodým derivative process, $\xi_t^{\tilde{\theta}}$, where $\tilde{\theta}_t \equiv \int_\Theta \theta_t \tilde{\eta}_t(\theta)\, d\theta$. Under $(\Omega, \mathscr{F}, \mathbb{Q}^{\tilde{\theta}})$, $B_t^{\tilde{\theta}} \equiv B_t - \int_0^t \tilde{\theta}_s ds$ is a standard BM.

3.2 Constructing the DM's Utility Process

This subsection extends Duffie and Epstein's (1992) SDU to the smooth ambiguity preferences by taking continuous-time limits of the discrete-time model in the previous section. Recall that the DM's recursive utility under the discrete-time setting is represented by equation (1), where the CE function in the dual theory is $m_t^{\mathrm{D}}(V_{t+h})$ in equation (4). Applying the implicit function theorem to $V_t = W\left(C_t, m_t^{\mathrm{D}}(V_{t+h})\right)$, $m_t^{\mathrm{D}}(V_{t+h})$ is represented by

$$m_t^{\mathrm{D}}(V_{t+h}) = H(C_t, V_t, h),$$

for some function $H : (R_+, R_+, (0,1)) \to R_+$ that depends on the time interval, h.

Then, the differential of the CE with respect to h evaluated at $h = 0$ can be expressed as follows:

$$\begin{aligned}\left.\frac{d}{dh} m_t^{\mathrm{D}}(V_{t+h})\right|_{h=0} &= \frac{\partial H}{\partial h}(C_t, V_t, 0) \\ &= -f(C_t, V_t),\end{aligned} \tag{5}$$

where $f(c,v) \equiv -\partial H(c,v,0)/\partial h$ is determined by the functional form of $W(\cdot,\cdot)$. Once $f(\cdot,\cdot)$ and $m_t^{\mathrm{D}}(\cdot)$ are given, we can construct the corresponding SDU process. The combination of these primitive functions, $\left(f, m_t^{\mathrm{D}}\right)$, is called the aggregator of the SDU. It is assumed that $f(\cdot,\cdot)$ is Lipschitz in utility; that is, there is a positive constant k such that $|f(c,v) - f(c,w)| \leq k|v-w|$ for all $(c,v,w) \in R_+^3$. It is also assumed that $f(\cdot,\cdot)$ satisfies a growth condition in consumption; that is, there are positive constants k_1 and k_2 such that $|f(c,0)| \leq k_1 + k_2 c$ for all $c \in R_+$.

Because we consider only Brownian uncertainty, we conjecture that the SDU process, $V = \{V_t\}_{t=0}^T$, has the following stochastic differential representation under the continuous-time setting:

$$
\begin{aligned}
dV_t &= \left(\mu_t + \sigma_t^\top \theta_t\right) dt + \sigma_t^\top dB_t^\theta && (6)\\
&= \left(\mu_t + \sigma_t^\top \tilde{\theta}_t\right) dt + \sigma_t^\top dB_t^{\tilde{\theta}}, && (7)
\end{aligned}
$$

where μ_t and σ_t are both $\mathscr{F}_t$-measurable. Given the aggregator, $\left(f, m_t^{\mathrm{D}}\right)$, the volatility process, $\sigma = \{\sigma_t\}_{t=0}^T$, of the SDU is endogenously determined.

First, we derive the continuous-time limit of the KP CE conditional on each $\mathbb{Q}^\theta$, represented in equation (3), by the following lemma.

Lemma 1. *Given the utility process in equation (6), the differential of time-t conditional KP CE, $m_t^\theta(V_{t+h})$, with respect to h is expressed by:*

$$
\left.\frac{d}{dh} m_t^\theta(V_{t+h})\right|_{h=0} = \mu_t + \sigma_t^\top \theta_t - \frac{1}{2}\mathfrak{R}(V_t)|\sigma_t|^2, \tag{8}
$$

where $\mathfrak{R}(V_t) \equiv -u''(V_t)/u'(V_t)$ is the DM's absolute risk aversion.

Proof. Using Ito's lemma, we find from equation (6) that

$$
u(V_{t+h}) = u(V_t) + \int_t^{t+h} \left\{ u'(V_s)\left(\mu_s + \sigma_s^\top \theta_s\right) + \frac{1}{2}u''(V_s)|\sigma_s|^2 \right\} ds + \int_t^{t+h} u'(V_s)\,\sigma_s^\top dB_s^\theta.
$$

Then, applying the Taylor expansion, the conditional KP CE can be approximated as

$$m_t^{\theta}\left(V_{t+h}\right) = V_t + \left\{\left(\mu_t + \sigma_t^{\top}\theta_t\right) - \frac{1}{2}\Re\left(V_t\right)\left|\sigma_t\right|^2\right\} h + o\left(h\right).$$

Because $m_t^{\theta}\left(V_t\right) = V_t$,

$$\begin{aligned}
\left.\frac{d}{dh} m_t^{\theta}\left(V_{t+h}\right)\right|_{h=0} &= \lim_{h\to 0} \frac{m_t^{\theta}\left(V_{t+h}\right) - m_t^{\theta}\left(V_t\right)}{h} \\
&= \lim_{h\to 0} \frac{\left\{\left(\mu_t + \sigma_t^{\top}\theta_t\right) - \frac{1}{2}\Re\left(V_t\right)\left|\sigma_t\right|^2\right\} h + o\left(h\right)}{h} \\
&= \mu_t + \sigma_t^{\top}\theta_t - \frac{1}{2}\Re\left(V_t\right)\left|\sigma_t\right|^2,
\end{aligned}$$

and we obtain equation (8). □

Using the conditional KP CE in equation (8), we can also derive the continuous-time limit of the time-t CE under the dual theory, represented in equation (4), by the following lemma.

Lemma 2. *The differential of the time-t CE under the dual theory, $m_t^{\mathrm{D}}\left(V_{t+h}\right)$, with respect to h is expressed by*

$$\left.\frac{d}{dh} m_t^{\mathrm{D}}\left(V_{t+h}\right)\right|_{h=0} = \mu_t + \sigma_t^{\top}\tilde{\theta}_t - \frac{1}{2}\Re\left(V_t\right)\left|\sigma_t\right|^2. \tag{9}$$

Proof. From equations (4), (8), and $m_t^{\mathrm{D}}\left(V_t\right) = V_t$, equation (9) is obtained directly as follows:

$$\begin{aligned}
\left.\frac{d}{dh} m_t^{\mathrm{D}}\left(V_{t+h}\right)\right|_{h=0} &= \lim_{h\to 0} \frac{m_t^{\mathrm{D}}\left(V_{t+h}\right) - m_t^{\mathrm{D}}\left(V_t\right)}{h} \\
&= \lim_{h\to 0} \frac{\int_{\Theta} m_t^{\theta}\left(V_{t+h}\right)\tilde{\eta}_t\left(\theta\right) d\theta - V_t}{h} \\
&= \lim_{h\to 0} \int_{\Theta} \left\{\frac{m_t^{\theta}\left(V_{t+h}\right) - V_t}{h}\right\} \tilde{\eta}_t\left(\theta\right) d\theta \\
&= \int_{\Theta} \left\{\left.\frac{d}{dh} m_t^{\theta}\left(V_{t+h}\right)\right|_{h=0}\right\} \tilde{\eta}_t\left(\theta\right) d\theta \\
&= \int_{\Theta} \left\{\mu_t + \sigma_t^{\top}\theta_t - \frac{1}{2}\Re\left(V_t\right)\left|\sigma_t\right|^2\right\} \tilde{\eta}_t\left(\theta\right) d\theta \\
&= \mu_t + \sigma_t^{\top}\tilde{\theta}_t - \frac{1}{2}\Re\left(V_t\right)\left|\sigma_t\right|^2,
\end{aligned}$$

where the fourth equality follows by Lebesgue's convergence theorem. □

Substituting equations (5) and (9) into equation (7), it can be seen that the SDU process, V_t, is the solution of the following backward stochastic differential equation (BSDE):

$$dV_t = -\left\{ f\left(C_t, V_t\right) - \frac{1}{2}\mathfrak{R}\left(V_t\right)\left|\sigma_t\right|^2 \right\} dt + \sigma_t^{\mathsf{T}} dB_t^{\tilde{\theta}}, \quad V_T = 0. \tag{10}$$

The BSDE (10) implies that the DM's time-t utility can be expressed as follows:

$$V_t = E_t^{\tilde{\theta}}\left[\int_t^T \left\{ f\left(C_s, V_s\right) - \frac{1}{2}\mathfrak{R}\left(V_s\right)\left|\sigma_s\right|^2 \right\} ds\right], \quad V_T = 0,$$

where $E_t^{\tilde{\theta}}[\cdot]$ is the conditional expectation under $\mathbb{Q}^{\tilde{\theta}}$. Finally, through the technique used by Duffie and Epstein (1992), we can derive an ordinally equivalent SDU process of V_t by the following proposition.

Proposition 2. *In regard to a given aggregator, $\left(f, m_t^{\mathrm{D}}\right)$, and the corresponding SDU process, V_t, define an alternative aggregator, $\left(\hat{f}, \hat{m}_t^{\mathrm{D}}\right)$, that satisfies*

$$m_t^{\mathrm{D}}\left(v\right) = u^{-1}\left[\hat{m}_t^{\mathrm{D}}\left(u\left(v\right)\right)\right], \tag{11}$$

$$f\left(c, v\right) = \frac{\hat{f}\left(c, u\left(v\right)\right)}{u'\left(v\right)}. \tag{12}$$

Assuming that $u(0) = 0$, an alternative SDU process, $\hat{V}_t \equiv u\left(V_t\right)$, corresponding to the aggregator, $\left(\hat{f}, \hat{m}_t^{\mathrm{D}}\right)$, is obtained through

$$\hat{V}_t = E_t^{\tilde{\theta}}\left[\int_t^T \hat{f}\left(C_s, \hat{V}_s\right) ds\right]. \tag{13}$$

Then, $\hat{V}_t$ is ordinally equivalent to V_t in the sense that $\hat{V}_t$ and V_t rank the same ordering among consumption processes.

Proof. Applying the Ito's formula to $\hat{V}_t = u\left(V_t\right)$ and from equation (7), we can see that $\hat{V}_t$ has the following stochastic differential representation:

$$d\hat{V}_t = \left\{ u'\left(V_t\right)\left(\mu_t + \sigma_t^{\mathsf{T}}\tilde{\theta}_t\right) + \frac{1}{2}u''\left(V_t\right)\left|\sigma_t\right|^2 \right\} dt + u'\left(V_t\right)\sigma_t^{\mathsf{T}} dB_t^{\tilde{\theta}}.$$

In addition, from equations (11) and (12), the differential of $\hat{m}_t^{\mathrm{D}}\left(\hat{V}_{t+h}\right)$ with respect to h is

$$\begin{aligned}
\left.\frac{d}{dh}\hat{m}_t^{\mathrm{D}}\left(\hat{V}_{t+h}\right)\right|_{h=0} &= u'\left(V_t\right)\left.\frac{d}{dh}m_t^{\mathrm{D}}\left(V_{t+h}\right)\right|_{h=0} \\
&= u'\left(V_t\right)\left\{\mu_t + \sigma_t^{\mathsf{T}}\tilde{\theta}_t - \frac{1}{2}\mathfrak{R}\left(V_t\right)\left|\sigma_t\right|^2\right\} \\
&= -\hat{f}\left(C_t, \hat{V}_t\right).
\end{aligned}$$

From these relationships, it can be seen that $\hat{V}_t$ is the solution of the following BSDE:

$$d\hat{V}_t = -\hat{f}\left(C_t, \hat{V}_t\right)dt + \hat{\sigma}_t^{\mathsf{T}} dB_t^{\tilde{\theta}}, \quad \hat{V}_T = 0,$$

where $\hat{\sigma}_t \equiv u'\left(V_t\right)\sigma_t$. This implies that $\hat{V}_t$ can be represented by equation (13). The equivalence between $\hat{V}_t$ and V_t results directly from the fact that $u(\cdot)$ is a strictly increasing function. □

Because of its tractability, we develop the following analysis by mainly utilizing the ordinally equivalent SDU process, $\hat{V}_t$. Accordingly, if there is no confusion, we only denote the ordinally equivalent SDU and the corresponding aggregator by V_t and $\left(f, m_t^{\mathrm{D}}\right)$, respectively.

We showed above that the SDU process for the smooth ambiguity preferences can be constructed similarly as the original SDU without ambiguity, where the only difference is the probability measure used to calculate expectation. Therefore, the SDU in this study takes over the properties of the original SDU; including, continuity, monotonicity for consumption, time consistency, and concavity, once $f(\cdot,\cdot)$ satisfies the adequate conditions provided by Duffie and Epstein (1992). In the following subsections, we assume that these conditions are satisfied.

3.3 Ambiguity Aversion

In this subsection, we formalize the DM's ambiguity aversion implied by the SDU in equation (13). First, by following Chen and Epstein (2002), we define an unambiguous event, $A \in \mathscr{F}$, as one whose probability agrees among all first-order probabilities, $\mathbb{Q}^{\theta} \in \Delta$.[7] Then, the class $\mathscr{U}$ of unambiguous events is defined as

$$\mathscr{U} = \left\{ A \in \mathscr{F} : \mathbb{Q}^{\theta}(A) = \mathbb{Q}^{\theta'}(A) \text{ for } \forall \theta, \theta' \in \Theta \right\}.$$

All other events are called ambiguous.

For the events class, $\mathscr{U}$, we refer to a process, $C^{\mathrm{ua}} = \{C_t^{\mathrm{ua}}\}_{t=0}^{T}$, as an unambiguous consumption process, if C_t^{ua} is $\mathscr{U}$-measurable for each $t \le T$. We then compare the degree of ambiguity aversion between different DMs by the following definition.

Definition 1. *Given the unambiguous events class $\mathscr{U}$, the utility V^* is more ambiguity-averse than V if*

$$V(C) \le V(C^{\mathrm{ua}}) \Rightarrow V^*(C) \le V^*(C^{\mathrm{ua}}),$$

holds for an arbitrary consumption process, C, and any unambiguous one, C^{ua}.

Definition 1 states that whenever a DM having utility V prefers an unambiguous consumption process, C^{ua}, over a possibly ambiguous one, C, then a more ambiguity-averse DM having utility V^* should also prefer the former over the latter. Note that we can compare the degree of DMs' ambiguity aversion under the fixed class of unambiguous events. Therefore, unlike the definition by Chen and Epstein (2002), we can manage the degree of the DM's ambiguity aversion separately from the degree of ambiguity that he/she subjectively perceives.

Given Definition 1, the following proposition confirms that the degree of the DM's ambiguity aversion is determined by the convexity of the transformation function, $\varphi(\cdot)$, in equation (4).

Proposition 3. *For fixed $f(\cdot,\cdot)$ and $u(\cdot)$, consider two CEs, $m_t^{\mathrm{D1}}(\cdot)$ and $m_t^{\mathrm{D2}}(\cdot)$, generated by transformation functions, $\varphi^1(\cdot)$ and $\varphi^2(\cdot)$, respectively. Denote the SDUs corresponding to each aggregator, (f, m_t^{D1}) and (f, m_t^{D2}), as V_t^1 and V_t^2. Then, V^2 is more ambiguity-averse than V^1 if there is some strictly increasing, continuous, and convex function $J(\cdot) : [0,1] \to [0,1]$ such that*

$$\varphi^2 = J(\varphi^1).$$

[7] Note that Δ includes the reference probability measure, $\mathbb{P}$.

Proof. We consider pre-normalized SDU processes, V_t^1 and V_t^2, corresponding to aggregators, (f, m_t^{D1}) and (f, m_t^{D2}), respectively, that satisfy the following BSDEs:

$$dV_t^i = \varsigma^i\left(V_t^i, \sigma_t^i, \omega, t\right)dt + \left(\sigma_t^i\right)^{\mathsf{T}} dB_t, \quad V_T^i = 0, \quad \text{for } i = 1, 2,$$

where

$$\varsigma^i\left(V_t^i, \sigma_t^i, \omega, t\right) = -\left\{f\left(C_t\left(\omega\right), V_t^i\right) + \left(\sigma_t^i\right)^{\mathsf{T}} \tilde{\theta}_t^i - \frac{1}{2}\Re\left(V_t^i\right)\left|\sigma_t^i\right|^2\right\},$$

and $\tilde{\theta}_t^i = \int_\Theta \theta_t \tilde{\eta}_t^i(\theta)\, d\theta$ is the time-t conditional expectation of θ_t under the distorted second-order probability, $\tilde{\eta}_t^i$, implied by the distorted decumulative distribution function, $\varphi^i[G_t(\cdot)]$. Obviously, $V^1(C^{\mathrm{ua}}) = V^2(C^{\mathrm{ua}})$ holds for an unambiguous consumption process, C^{ua}.

Meanwhile, for a given decumulative function, $G_t(\cdot) : R_+ \to [0,1]$, we can obtain the following relation from the convexity of $J(\cdot)$:

$$\varphi^2\left[G_t(z)\right] = J\left(\varphi^1\left[G_t(z)\right]\right) \le \varphi^1\left[G_t(z)\right], \quad \text{for } \forall z \in (0, \infty).$$

Therefore, from equation (4), $m_t^{\mathrm{D2}}(V_{t+h}) \le m_t^{\mathrm{D1}}(V_{t+h})$ holds for some common V_{t+h}. By noting $m_t^{\mathrm{D1}}(V_t) = m_t^{\mathrm{D2}}(V_t) = V_t$, this implies that

$$\begin{aligned}
\frac{d}{dh} m_t^{\mathrm{D2}}(V_{t+h})\bigg|_{h=0} &= \mu_t + \sigma_t^{\mathsf{T}} \tilde{\theta}_t^2 - \frac{1}{2}\Re(V_t)\left|\sigma_t\right|^2 \\
&\le \mu_t + \sigma_t^{\mathsf{T}} \tilde{\theta}_t^1 - \frac{1}{2}\Re(V_t)\left|\sigma_t\right|^2 \\
&= \frac{d}{dh} m_t^{\mathrm{D1}}(V_{t+h})\bigg|_{h=0} \\
\therefore \sigma_t^{\mathsf{T}} \tilde{\theta}_t^2 &\le \sigma_t^{\mathsf{T}} \tilde{\theta}_t^1,
\end{aligned}$$

where σ_t is the volatility process of V_t.

The above inequality in turn implies that $\varsigma^2(V_t, \sigma_t, \omega, t) \ge \varsigma^1(V_t, \sigma_t, \omega, t)$. Then, by El Karoui et al.'s (1997) comparison theorem for BSDE (Theorem 2.2 in their study), $V_t^1 \ge V_t^2$ for almost every $t \in [0, T]$. This implies that $V^1(C) \ge V^2(C)$ holds for an arbitrary consumption process, C, so that V^2 is more ambiguity-averse than V^1. □

Proposition 3 shows that our model enables us to change the degree of the DM's ambiguity aversion without affecting his/her attitude toward intertemporal substitution or risk. That is, we can achieve a three-way separation of the DM's preferences: intertemporal substitution, risk aversion, and ambiguity aversion.

Next, we provide the condition under which a given SDU, V_t, is ambiguity averse in an absolute sense. Therefore, we consider a consumption process, $C = \{C_t\}_{t=0}^T$, having the following stochastic differential representation:

$$dC_t = \left(\mu_{C,t} + \sigma_{C,t}^\top \theta_t\right) dt + \sigma_{C,t}^\top dB_t^\theta, \quad C_0 = c, \quad \text{for } \theta \in \Theta. \tag{14}$$

If there are multiple $\theta = \{\theta_t\}_{t=0}^T$ in Θ, then C is an ambiguous consumption process. For the consumption process in equation (14), we define the corresponding unambiguous consumption process, denoted by $\overline{C}^{\text{ua}}$, whose stochastic differential representation is

$$d\overline{C}_t^{\text{ua}} = \left(\mu_{C,t} + \sigma_{C,t}^\top \overline{\theta}_t\right) dt + \sigma_{C,t}^\top dB_t, \quad \overline{C}_0^{\text{ua}} = c. \tag{15}$$

Note that both the expected increment and volatility of $\overline{C}^{\text{ua}}$ under the reference probability measure, $\mathbb{P}$, are identical to those of C under the compound probability measure, $\mathbb{Q}^{\overline{\theta}}$. With these notations, we define the DM's ambiguity aversion as follows.

Definition 2. *For the consumption processes in equations (14) and (15), the DM is ambiguity averse if his/her SDU, V, satisfies*

$$V(C) \leq V(\overline{C}^{\text{ua}}).$$

This would be a natural definition to describe the DM's ambiguity aversion in our setting. From equations (14) and (15), the unconditional moments of C_t under $\mathbb{Q}^{\overline{\theta}}$ and those of $\overline{C}_t^{\text{ua}}$ coincide for every $t \in [0,T]$. Under this condition, Definition 2 states that an ambiguity-averse DM would never prefer the possibly ambiguous consumption process, C, over the unambiguous consumption process, $\overline{C}^{\text{ua}}$. From this definition, we can consider the DM's ambiguity attitude in the absolute sense and the following proposition provides the condition under which the DM is ambiguity averse.

Proposition 4. *For fixed $f(\cdot,\cdot)$ and $u(\cdot)$, consider the CE, $m_t^{\mathrm{D}}(\cdot)$, generated by a transformation function $\varphi(\cdot)$. If $\varphi(\cdot)$ is a convex function, then the SDU corresponding to the aggregator, $\left(f, m_t^{\mathrm{D}}\right)$, is ambiguity averse.*

Proof. If a transform function, $\varphi^1(\cdot)$, is a linear (hence identity) function, then $\varphi^1[G_t(z)] = G_t(z)$ holds for any decumulative function, $G_t(\cdot)$. In this case, the DM's time-t CE is the KP CE under $\mathbb{Q}^{\overline{\theta}}$; that is, the aggregator $(f, m_t^{\overline{\theta}})$ derives the SDU, V^1, that satisfies the following:

$$\begin{aligned} V^1(C) &= E_0^{\overline{\theta}}\left[\int_0^T f(C_s, V_s)\, ds\right] \\ &= E_0\left[\int_0^T f\left(\overline{C}_s^{\mathrm{ua}}, V_s\right) ds\right] \\ &= V^1\left(\overline{C}^{\mathrm{ua}}\right). \end{aligned}$$

Meanwhile, for a given convex transform function, $\varphi^2(\cdot)$, there should be some strictly increasing, continuous, and convex function, $J(\cdot): [0,1] \to [0,1]$, such that $\varphi^2 = J(\varphi^1)$. We denote the SDU corresponding to (f, m_t^{D2}) by V^2, where $m_t^{\mathrm{D2}}(\cdot)$ is the time-t CE generated by $\varphi^2(\cdot)$. Then, from Proposition 3, V^2 is more ambiguity averse than V^1, which implies that $V^2(C) \leq V^2(\overline{C}^{\mathrm{ua}})$. □

4 Application to Asset Pricing

In this section, we illustrate how our model can be applied to asset pricing. Under the discrete-time setting, Ju and Miao (2012) apply the original KMM model to asset pricing and show that KMM preferences can help resolve many asset-pricing puzzles, including the equity premium puzzle, the equity-volatility puzzle, and the risk-free rate puzzle. Their model can also replicate counter-cyclical variations in equity premium and equity volatility. For a comparison between the original KMM model and our dual theory, we derive asset prices under a setting that is a continuous-time analogue of Ju and Miao's model. We then examine whether our asset pricing result is compatible with theirs.

4.1 The Economy

We consider an endowment economy inhabited by an infinitely lived representative agent. Under a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{Q}^{\theta^*})$, the aggregate consumption, $C = \{C_t\}_{t=0}^{\infty}$, and the equity dividend, $D = \{D_t\}_{t=0}^{\infty}$, are governed by the following stochastic process:

$$\begin{aligned} \frac{dY_t}{Y_t} &\equiv \left(\frac{dC_t}{C_t}, \frac{dD_t}{D_t}\right)^{\top} \\ &= (\delta + \kappa\theta_t^*)\,dt + \kappa dB_t^{\theta^*}, \end{aligned} \tag{16}$$

where $B_t^{\theta^*}$ is a two-dimensional standard BM under $\mathbb{Q}^{\theta^*}$. $\delta = (\delta_C\ \delta_D)^{\top}$ is a constant 2×1 vector and κ is a constant 2×2 matrix. The first row of κ is denoted by κ_C, while the second row of κ is denoted by κ_D. θ_t^* is a two-dimensional $\mathscr{F}_t$ adopted process.

In this economy, θ_t^* takes one of two possible outcomes, $\theta_t^* \in \{\theta^1, \theta^2\}$, that follows the two-states continuous-time Markov chain, also defined on $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{Q}^{\theta^*}\right)$ with the infinitesimal generator matrix

$$\Lambda = \begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{pmatrix}, \quad \lambda_{12}, \lambda_{21} > 0.$$

Intuitively, the transition probability of θ_t^* from θ^i to the other θ^j in infinitesimal time is $\lambda_{ij}dt$. It is assumed that the evolution of θ_t^* is independent of $B_t^{\theta^*}$.

In terms of ambiguity, the representative agent cannot observe θ_t^*, and his/her information set at time t is generated by the observed fundamentals process up to t, denoted by $\mathscr{F}_t^Y \subset \mathscr{F}_t$. Therefore, from the agent's point of view, there are two alternative first-order probability measures, $\Delta = \{\mathbb{Q}^{\theta^1}, \mathbb{Q}^{\theta^2}\}$, under each of which the fundamentals process is expressed as follows:

$$\frac{dY_t}{Y_t} = \left(\delta + \kappa\theta^i\right)dt + \kappa dB_t^{\theta^i}, \quad \text{for } i = 1, 2.$$

where $B_t^{\theta^i}$ is a two-dimensional standard BM under $(\mathscr{F}^Y, \mathbb{Q}^{\theta^i})$. It is assumed that $\kappa_C\theta^1 > \kappa_C\theta^2$. That is, $\mathbb{Q}^{\theta^1}$ is a more favorable probability measure for the agent than $\mathbb{Q}^{\theta^2}$. Without loss of generality, we set

$\theta^1 = (0,0)^\top$ so that $\mathbb{Q}^{\theta^1}$ is selected as the reference probability measure, i.e., $\mathbb{Q}^{\theta^1} = \mathbb{P}$.

With this ambiguity, the agent subjectively assigns the second-order probabilities for each θ^i. We denote the time-t second-order probability for each θ^i by $\eta_t^i \equiv \eta_t\left(\theta^i\right)$ and represent the time-t conditional expectation of θ_t^* by $\overline{\theta}_t \equiv \eta_t^2\theta^2$. Using the $\mathscr{F}_t^Y$ adapted process $\overline{\theta}_t$, we can construct the compound probability measure, $\mathbb{Q}^{\overline{\theta}}$, through the Radon-Nikodým derivative process with respect to the reference measure, $\xi_t^{\overline{\theta}}$. In particular, if the agent is a Bayesian, the stochastic process of η_t^i can be obtained through the standard filtering theory.

Lemma 3. *Given the true fundamental process of equation (16) and a prior probability, η_0^1, the updated Bayesian second-order probability η_t^1 is governed by the following process:*

$$d\eta_t^1 = \left\{\lambda_{21} - (\lambda_{12}+\lambda_{21})\,\eta_t^1\right\}dt - \eta_t^1\left(1-\eta_t^1\right)\left(\theta^2\right)^\top dB_t^{\overline{\theta}}, \tag{17}$$

where $B_t^{\overline{\theta}}$ is a two-dimensional standard BM under $\{\mathscr{F}^Y, \mathbb{Q}^{\overline{\theta}}\}$. The second-order probability for θ^2 is obtained by $\eta_t^2 = 1 - \eta_t^1$ for $\forall t \in [0,\infty]$.

Proof. See, for example, Liptser and Shiryaev (2001). □

4.2 The Agent's Preferences

Next, we assume the following transform function

$$\varphi(g) = g^{\alpha+1},$$

to represent the agent's ambiguity attitude. The DM is ambiguity averse if $\alpha \geq 0$, and the value of α represents the degree of the DM's ambiguity aversion. In this case, the time-t distorted second-order probabilities are $\tilde{\eta}_t^1 = \left(\eta_t^1\right)^{\alpha+1}$ and $\tilde{\eta}_t^2 = 1 - \tilde{\eta}_t^1$.

Under the distorted second-order probability, the time-t conditional expectation of θ_t^* is calculated as $\tilde{\theta}_t = \tilde{\eta}_t^2\theta^2$. The $\mathscr{F}_t^Y$ adapted process $\tilde{\theta}_t$ generates the distorted compound probability measure, $\mathbb{Q}^{\tilde{\theta}}$, through the following Radon-Nikodým derivative process with respect to the reference measure:

$$\xi_t^{\tilde{\theta}} = \exp\left\{-\frac{1}{2}\int_0^t \left|\tilde{\theta}_s\right|^2 ds + \int_0^t \tilde{\theta}_s dB_s\right\}, \quad \xi_0^{\tilde{\theta}} = 1. \tag{18}$$

Then, $B_t^{\tilde{\theta}} \equiv B_t - \int_0^t \tilde{\theta}_s ds$ is a two-dimensional standard BM under $\{\mathscr{F}^Y, \mathbb{Q}^{\tilde{\theta}}\}$. Using the relation that $B_t^{\tilde{\theta}} = B_t^{\theta^*} - \int_0^t \left(\tilde{\theta}_s - \theta_s^*\right) ds = B_t^{\overline{\theta}} - \int_0^t \left(\tilde{\theta}_s - \overline{\theta}_s\right) ds$, we can rewrite the stochastic processes in equations (16) and (17) as follows:

$$\frac{dY_t}{Y_t} = \left(\delta + \kappa\tilde{\theta}_t\right) dt + \kappa dB_t^{\tilde{\theta}}, \tag{19}$$

$$d\eta_t^1 = \zeta\left(\eta_t^1\right) dt + \chi\left(\eta_t^1\right) dB_t^{\tilde{\theta}}, \tag{20}$$

where

$$\zeta\left(\eta_t^1\right) = \lambda_{21} - \left(\lambda_{12} + \lambda_{21}\right)\eta_t^1 - \eta_t^1\left(1 - \eta_t^1\right)\left(\theta^2\right)^{\mathsf{T}}\left\{\tilde{\theta}_t - \overline{\theta}_t\right\},$$
$$\chi\left(\eta_t^1\right) = -\eta_t^1\left(1 - \eta_t^1\right)\left(\theta^2\right)^{\mathsf{T}}.$$

For the agent's SDU process, we assume that the normalized aggregator, $\left(f, m_t^{\mathrm{D}}\right)$, has the following form:[8]

$$f(c,v) = \left(\frac{\beta}{1-\rho}\right)\frac{c^{1-\rho} - \left\{(1-\gamma)v\right\}^{\frac{1-\rho}{1-\gamma}}}{\left\{(1-\gamma)v\right\}^{\frac{\gamma-\rho}{1-\gamma}}},$$
$$m_t^{\mathrm{D}}(w) = E_t^{\tilde{\theta}}(w).$$

$\beta \geq 0$ represents the subjective discount rate of the agent. $\gamma \geq 0$ represents the agent's relative risk aversion (RRA), while $\rho \geq 0$ represents the inverse of the agent's intertemporal elasticity of substitution (IES). When $\gamma = \rho$, the corresponding SDU is reduced to the time-additive power utility.

Although the explicit expression for the corresponding SDU process, V_t, cannot be obtained, its numerical solution is easily found by the following proposition.

[8] While this functional form of $f(\cdot,\cdot)$ violates the Lipschitz condition for utility, the existence and uniqueness of the SDU process are discussed in Duffie and Lions (1992) and Schroder and Skiadas (1999).

Proposition 5. *In this economy, the agent's SDU process, V_t, is expressed by*

$$V_t = A\left(\eta_t^1\right)\frac{C_t^{1-\gamma}}{1-\gamma}, \tag{21}$$

where $A(\cdot)$ is a solution of the following ordinal differential equation (ODE):

$$0 = \beta\left(\frac{1-\gamma}{1-\rho}\right)\{A(\eta)\}^{\frac{\rho-\gamma}{1-\gamma}} + (1-\gamma)\left\{\delta_C + \kappa_C\tilde{\theta}(\eta) - \frac{1}{2}\gamma|\kappa_c|^2 - \frac{\beta}{1-\rho}\right\}A(\eta) + \left\{\zeta(\eta) + (1-\gamma)\kappa_C\chi(\eta)^\top\right\}A'(\eta) + \frac{1}{2}|\chi(\eta)|^2 A''(\eta). \tag{22}$$

Proof. From the homotheticity of $f(c,v)$, we first guess that the functional form of V_t is represented by equation (21). Then, from equation (13), we obtain the following expression for V_t:

$$\begin{aligned} 0 &= f(C_t,V_t)\,dt + E_t^{\tilde{\theta}}(dV_t) \\ &= \left[\left(\frac{\beta}{1-\rho}\right)\frac{1-\left\{A\left(\eta_t^1\right)\right\}^{\frac{1-\rho}{1-\gamma}}}{\left\{A\left(\eta_t^1\right)\right\}^{\frac{\gamma-\rho}{1-\gamma}}}C_t^{1-\gamma}\right]dt + E_t^{\tilde{\theta}}(dV_t). \end{aligned} \tag{23}$$

Meanwhile, applying the Ito's formula to V_t in equation (21), it can be seen that V_t has the following stochastic differential expression:

$$dV_t = C_t^{1-\gamma}\left\{A\left(\eta_t^1\right)\frac{dC_t}{C_t} - \frac{1}{2}\gamma A\left(\eta_t^1\right)\left(\frac{dC_t}{C_t}\right)^2 + \frac{A'\left(\eta_t^1\right)}{1-\gamma}d\eta_t^1 + \frac{1}{2}\frac{A''\left(\eta_t^1\right)}{1-\gamma}\left(d\eta_t^1\right)^2 + A'\left(\eta_t^1\right)\frac{dC_t}{C_t}\cdot d\eta_t^1\right\}.$$

Substituting this expression and equations (19) and (20) into equation (23), we obtain the ODE (22). □

4.3 Asset Pricing

This subsection derives asset prices in our economy. The economy contains two assets: a risk-free asset with zero net supply and equity. The risk-free rate and equity price at time t are denoted by r_t and S_t, respectively. To facilitate asset pricing, we define an adapted process, $\pi_t^{\tilde{\theta}}$, as the stochastic discount factor (SDF) under $\mathbb{Q}^{\tilde{\theta}}$, which relates the time-t price of an arbitrary asset, denoted by p_t, and its continuation dividend sequence, $z = \{z_s\}_{s=t}^{\infty}$, through

$$\pi_t^{\tilde{\theta}} p_t = E_t^{\tilde{\theta}} \left[\int_t^{\infty} \pi_s^{\tilde{\theta}} z_s ds \right].$$

Following Duffie and Skiadas (1994), the SDF in this economy has the following form:

$$\pi_t^{\tilde{\theta}} = \exp\left\{ \int_0^t f_V (C_s, V_s) \, ds \right\} f_C (C_t, V_t),$$

where $f_C(c,v)$ and $f_V(c,v)$ represent partial derivatives of $f(c,v)$ for c and v, respectively. Then, applying the Ito's formula to $\pi_t^{\tilde{\theta}}$ and using equations (19), (20), and (21), its stochastic process is obtained as follows:

$$-\frac{d\pi_t^{\tilde{\theta}}}{\pi_t^{\tilde{\theta}}} = \left\{ \beta + \rho \left(\delta_C + \kappa_C \tilde{\theta}_t \right) - \frac{1}{2}\gamma(\gamma+1) |\kappa_C|^2 + (\gamma - \rho) L\left(\eta_t^1\right) \right\} dt + \left\{ \gamma\kappa_C + \left(\frac{\gamma-\rho}{1-\gamma} \right) \frac{A'\left(\eta_t^1\right)}{A\left(\eta_t^1\right)} \chi\left(\eta_t^1\right) \right\} dB_t^{\tilde{\theta}}, \quad (24)$$

where

$$L(\eta) = \delta_C + \kappa_C \tilde{\theta}(\eta) + \frac{\beta}{1-\rho} \left[\{A(\eta)\}^{-\frac{1-\rho}{1-\gamma}} - 1 \right] + \frac{1}{1-\gamma} \left\{ \zeta(\eta) - \gamma\kappa_C \chi(\eta)^{\top} \right\} \frac{A'(\eta)}{A(\eta)} - \frac{|\chi(\eta)|^2}{2(1-\gamma)} \left[\left(\frac{1-\rho}{1-\gamma} \right) \left\{ \frac{A'(\eta)}{A(\eta)} \right\}^2 - \frac{A''(\eta)}{A(\eta)} \right].$$

It is well known that the drift term of equation (24) is equal to the equilibrium risk-free rate, whereas the diffusion term of equation (24) is equal to the equilibrium market price of uncertainty under $\mathbb{Q}^{\tilde{\theta}}$. Using these facts, we obtain asset prices in this economy through the following proposition.

Proposition 6. *From the SDF process in equation (24), the equilibrium risk-free rate is given by*

$$r_t = \beta + \rho \left(\delta_C + \kappa_C \overline{\theta}_t \right) - \frac{1}{2}\gamma(\gamma+1) |\kappa_C|^2 + (\gamma - \rho) L\left(\eta_t^1\right) - \rho\kappa_C \left(\overline{\theta}_t - \tilde{\theta}_t \right). \quad (25)$$

Meanwhile, the equilibrium equity price is given by

$$S_t = F\left(\eta_t^1\right) D_t, \tag{26}$$

where $F(\cdot)$ is a solution of the following ODE:

$$\begin{aligned} -1 = F(\eta) &\left[\delta_D + \kappa_D \tilde{\theta}(\eta) - \kappa_D \left\{\gamma \kappa_C + \left(\frac{\gamma-\rho}{1-\gamma}\right)\frac{A'(\eta)}{A(\eta)}\chi(\eta)\right\}^{\top} - r(\eta)\right] \\ &+ F'(\eta)\left[\zeta(\eta) + \chi(\eta)\left\{\kappa_D - \gamma\kappa_C - \left(\frac{\gamma-\rho}{1-\gamma}\right)\frac{A'(\eta)}{A(\eta)}\chi(\eta)\right\}^{\top}\right] + \frac{1}{2}F''(\eta)\left|\chi(\eta)\right|^2. \end{aligned} \tag{27}$$

Proof. From the drift term of the SDF process in equation (24), the risk-free rate in equation (25) is obtained immediately. As to the equity price, we conjecture that S_t has the form of equation (26). Then, applying the Ito's formula to S_t, the equity return process is given by

$$\frac{dS_t}{S_t} + \frac{D_t}{S_t}dt = \delta_S\left(\eta_t^1\right)dt + \kappa_S\left(\eta_t^1\right)dB_t^{\tilde{\theta}},$$

where

$$\delta_S\left(\eta_t^1\right) = \delta_D + \kappa_D\tilde{\theta}_t + \frac{F'\left(\eta_t^1\right)}{F\left(\eta_t^1\right)}\left\{\zeta\left(\eta_t^1\right) + \kappa_D\chi\left(\eta_t^1\right)^{\top}\right\} + \frac{1}{2}\frac{F''\left(\eta_t^1\right)}{F\left(\eta_t^1\right)}\left|\chi\left(\eta_t^1\right)\right|^2 + \frac{1}{F\left(\eta_t^1\right)}, \tag{28}$$

$$\kappa_S\left(\eta_t^1\right) = \kappa_D + \frac{F'\left(\eta_t^1\right)}{F\left(\eta_t^1\right)}\chi\left(\eta_t^1\right). \tag{29}$$

Because the diffusion term of equation (24) represents the market price of uncertainty, the time-t equity premium under $\mathbb{Q}^{\tilde{\theta}}$ is calculated by

$$\delta_S\left(\eta_t^1\right) - r_t = \kappa_S\left(\eta_t^1\right)\left\{\gamma\kappa_C + \left(\frac{\gamma-\rho}{1-\gamma}\right)\frac{A'\left(\eta_t^1\right)}{A\left(\eta_t^1\right)}\chi\left(\eta_t^1\right)\right\}^{\top}.$$

Substituting equations (28) and (29) into the above expression and rearranging terms, we obtain the ODE (27). □

From equation (25), we can decompose the risk-free rate into three components. The first three terms on the right side, $\beta+\rho\left(\delta_C+\kappa_C\overline{\theta}_t\right)-\frac{1}{2}\gamma(\gamma+1)|\kappa_C|^2$, are the sum of the agent's subjective discount rate, intertemporal substitution of consumption, and precautionary saving rate. If the agent's utility is time-additive (i.e., $\gamma=\rho$) and he/she is ambiguity neutral (i.e., $\tilde{\theta}_t=\overline{\theta}_t$), the risk-free rate would be determined by only these three terms. The fourth term in equation (25), $(\gamma-\rho)L\left(\eta_t^1\right)$, captures the effect of separation of the agent's risk attitude from his/her intertemporal attitude. The last term in equation (25), $-\rho\,\kappa_C\left(\overline{\theta}_t-\tilde{\theta}_t\right)$, represents the effect of the agent's ambiguity aversion on the risk-free rate. It shows that the agent's ambiguity aversion (i.e.,$\tilde{\theta}_t\leq\overline{\theta}_t$) lowers the risk-free rate in equilibrium. Because an ambiguity-averse agent behaves like a pessimist, he/she saves more by consumption smoothing.

The equity premium depends on the probability measure. Because we want to examine the effect of the agent's ambiguity aversion on the equity premium, it would be natural to express that effect under the compound probability measure, $\mathbb{Q}^{\overline{\theta}}$, instead of $\mathbb{Q}^{\tilde{\theta}}$. Accordingly, we express the market price of uncertainty under $\mathbb{Q}^{\overline{\theta}}$ by the following proposition.

Proposition 7. *The time-t market price of uncertainty under $\mathbb{Q}^{\overline{\theta}}$, denoted by $\overline{\phi}_t=\overline{\phi}\left(\eta_t^1\right)$, is given by*

$$\overline{\phi}\left(\eta_t^1\right)=\gamma\kappa_C+\left(\frac{\gamma-\rho}{1-\gamma}\right)\frac{A'\left(\eta_t^1\right)}{A\left(\eta_t^1\right)}\chi\left(\eta_t^1\right)+\left(\overline{\theta}_t-\tilde{\theta}_t\right). \tag{30}$$

Proof. Define $\pi_t^{\overline{\theta}}\equiv\pi_t^{\tilde{\theta}}\xi_t^{\tilde{\theta}}/\xi_t^{\overline{\theta}}$ as the SDF under $\mathbb{Q}^{\overline{\theta}}$. Applying the Ito's formula to $\pi_t^{\overline{\theta}}$ and using equations (18), and (24), its stochastic differential representation is obtained as

$$-\frac{d\pi_t^{\overline{\theta}}}{\pi_t^{\overline{\theta}}}=r_t dt+\overline{\phi}\left(\eta_t^1\right)^{\top}dB_t^{\overline{\theta}},$$

where r_t and $\overline{\phi}(\cdot)$ are equal to equation (25) and equation (30), respectively. Therefore, the stochastic process $\overline{\phi}_t$ in equation (30) serves as the market price of uncertainty under $\mathbb{Q}^{\overline{\theta}}$. ☐

Proposition 7 indicates that we can also decompose the equity premium into three components. The first term on the right side of equation (30) corresponds to the market price of risk when the agent's utility is time-additive. The second term represents the premium for the late resolution of uncertainty in equity. When $\gamma>\rho$, the agent prefers an earlier resolution of uncertainty; therefore, he/she hesitates to hold equity whose payoffs depend on the persistent fluctuation in θ_t. The third term represents the ambiguity premium. As the agent is more ambiguity averse, he/she requires more premium for equity that yields ambiguous dividends.

4.4 Calibration

In this subsection, we calibrate our continuous-time model and compare the asset pricing result with that in the study by Ju and Miao (2012). They use the U.S. annual data set of real returns for the S&P 500 and the six-month commercial paper from 1871 to 1993, which was originally examined by Cecchetti et al. (2000). In their sample period, the historical averages of the risk-free rate which were approximated by the returns on the six-month commercial paper and the equity premiums of the S&P 500 are 2.66% and 5.75%, respectively. Meanwhile, the historical standard deviations of risk-free rate and equity premiums are 5.13% and 19.02%, respectively. These unconditional moments of asset returns are the main targets of our calibration.

Table 1 presents the parameter values used to simulate the fundamentals process. We select each value so that the moments of our fundamentals process match those in the study by Ju and Miao (2012). The values of λ_{12} and λ_{21} imply that θ_t^* is highly persistent in θ^1. If $\theta_t^* = \theta^1$, the probability of θ_t^* having the same value more than one year is $e^{-0.031} = 0.97$. Meanwhile, if $\theta_t^* = \theta^2$, this probability is only $e^{-0.675} = 0.51$. The unconditional probability of θ_t^* being θ^1 is calculated by $\lambda_{21}\left(\lambda_{12}+\lambda_{21}\right)^{-1} = 0.96$. The parameter values in Table 1 also imply that the expected log growth rates of consumption and dividends are 2.3% and 3.0%, respectively, under $\mathbb{Q}^{\theta^1}$, but -6.8% and -21.8%, respectively, under $\mathbb{Q}^{\theta^2}$. Therefore, the unconditional expected log growth rates of consumption and dividends are both approximately 1.9%. From the values of κ_C and κ_D, the standard deviations of log growth rates of consumption and dividends are 3.1% and 12.0%, respectively, and the instantaneous correlation between the two rates is 0.72.

For the agent's preferences parameters, Ju and Miao (2012) set $(\rho, \gamma) = (1/1.5, 2)$ for their recursive utility. They use the KMM CE in equation (2) with the following form of the second-order utility:

$$v(m) = \frac{m^{1-\psi}}{1-\psi}.$$

In their model, the parameter ψ determines the degree of the agent's ambiguity aversion. They set $(\beta, \psi) = (0.025, 8.864)$ to match the first moments of the risk-free rate and equity premium. We set the same values of

(ρ,γ) and choose other parameter values by $(\beta,\alpha)=(0.023,1.3)$ to match the first moments of the risk-free rate and equity premium.

In our simulation, we discretize our model by Euler approximation. The time interval $\Delta t=0.001$ is set to generate a sequence of 1,000,000 (1,000 years) artificial fundamentals. We then derive asset prices through Proposition 6 and calculate annualized returns of risk-free assets and equities.

Table 2 presents our simulation results. The third row indicates the simulation result for our baseline case. With these parameter values, our model nearly completely replicates the unconditional simulated moments of the asset returns in Ju and Miao's baseline case. On the other hand, the degree of agent's ambiguity aversion can be interpreted more intuitively in our model through the value of $E(\tilde{\eta}^1)$. In our baseline case, the agent decreases the second-order probability for $\mathbb{Q}^{\theta^1}$ from 0.96 to 0.92 on average. This degree of agent pessimism is sufficient to replicate the U.S. historical moments of asset returns.

The third to fifth rows examine the effect of the agent's ambiguity aversion, α, on asset returns. Comparing simulated moments among rows, it can be seen that α plays a critical role in our asset pricing. In particular, when the agent is ambiguity neutral (i.e., $\alpha=0$), our model cannot replicate the U.S. historical equity premium at all. When we shift the agent's preferences from the SDU to the time-additive power utility (see the changes between the fifth and sixth rows), the asset pricing results worsen. The increase in the agent's RRA, γ, or in his/her ambiguity-aversion, α, has little effect in improving the asset pricing results under the time-additive power utility (see the changes from the sixth to the seventh and eighth rows, respectively). This result suggests that the separation of the agent's RRA from IES combined with his/her ambiguity aversion is crucial in our model.

Besides the unconditional moments of asset returns, Ju and Miao's model also succeeds in explaining the dynamic phenomena of asset-return moments, including counter-cyclical variations in equity premium and equity volatility. As a continuous-time counterpart of their model, our model should also replicate these dynamic properties. To confirm this, Figure 2 presents the equity premium (left panel) and the equity volatility (right panel) conditional on the second-order probability, η_t^1. As can be seen from the figure, both conditional moments are increasing in the degree of the agent's ambiguity aversion. Furthermore, the peaks of both curves shift toward the right side as the degree of the agent's ambiguity aversion increases. When the economy is in a good state (i.e., $\theta_t^*=\theta^1$), η_t^1 stays at or near one, and both conditional moments are relatively low. However, when θ_t^* switches from θ^1 to θ^2, η_t^1 decreases and both conditional moments increase. In particular, the conditional moments increase more sharply as the agent becomes more ambiguity averse. Therefore, our model can also capture counter-cyclical variations in equity premium and equity volatility.

5 Conclusion

In this study, we apply Yaari's (1987) dual theory to the KMM model, and succeed in representing the DM's smooth ambiguity preferences under a continuous-time setting. The calibration results of asset pricing indicates that our continuous-time model replicates most properties of asset prices implied by Ju and Miao's (2012) discrete-time model. Therefore, our model can be considered as a continuous-time counterpart of the original KMM preferences. Using the techniques in this study, we can combine the smooth ambiguity preferences with powerful tools developed in continuous-time financial studies, including optimal consumption-portfolio choice, equilibrium asset pricing, and derivative pricing.

Meanwhile, Yaari's (1987) dual theory is not a complete "dual" of the SEU theory. Indeed, the dual theory is derived by replacing the independence axiom postulated in the SEU theory with the so-called dual independence axiom. From this perspective, the DM's behavior under the dual theory is not necessarily identical to that under the original SEU theory. The same argument should be true of the relation between our model and the KMM model. Therefore, the consequence of this difference should be examined more carefully, and this task will be a future focus of our research.

While this study concentrates on Brownian uncertainty, it would be straightforward to incorporate jump processes into our model. Because it is very difficult to determine the distributions of rarely occurring events, a model including jump processes might be more suited to represent the ambiguity surrounding the DM than a model including only Brownian uncertainty. Another objective of our future research is to incorporate jump processes into our continuous-time smooth ambiguity preferences model.

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δ_C	δ_D	κ_C	κ_D	$(\theta^1)^\top$	$(\theta^2)^\top$	λ_{12}	λ_{21}
0.024	0.038	$(0.031, 0)$	$(0.086, 0.084)$	$(0, 0)$	$(-2.935, 0.029)$	0.031	0.675

Table 1: Parameter values of the fundamentals process used in our calibration.

Unconditional Moments	$E(r)$	$\sigma(r)$	$E(R^{\text{eq}})$	$\sigma(R^{\text{eq}})$	$E(R^{\text{eq}})/\sigma(R^{\text{eq}})$	$E\left(\tilde{\eta}^1\right)$
U.S. Historical (1871-1993)	2.66	5.13	5.75	19.02	0.30	-
Ju and Miao (2012)	2.66	1.16	5.75	18.26	0.31	-
Our Model						
(ρ, γ, α)						
(1/1.5,2,1.3)	2.66	1.22	5.75	18.37	0.31	92.4
(1/1.5,2,0.5)	3.18	1.04	1.84	16.10	0.11	94.2
(1/1.5,2,0)	3.46	0.87	0.49	14.41	0.03	95.6
(2,2,0)	6.05	2.37	0.25	13.60	0.02	95.6
(3.5,3.5,0)	8.54	4.14	0.61	13.06	0.05	95.6
(3.5,3.5,1.3)	7.53	5.51	0.62	12.31	0.05	92.4

Table 2: Simulated moments of annualized asset returns. $E(\cdot)$ and $\sigma(\cdot)$ represent the unconditional mean and standard deviation of the simulated moments, respectively. R^{eq} represents the equity-premium. Except for $E(R^{\text{eq}})/\sigma(R^{\text{eq}})$, the Sharpe-ratio of equity return, each number represents a percentage. The first row shows the historical moments of U.S. asset returns from 1871 to 1993 estimated by Cecchetti et al. (2000). The second row shows the simulated moments in the study by Ju and Miao (2012) with the parameter values of $(\beta, \rho, \gamma, \psi) = (0.025, 1/1.5, 2.0, 8.864)$. The third to eighth rows show the simulated moments using our model according to various parameter values of (ρ, γ, α). In our simulations, the value of β is fixed at 0.023.

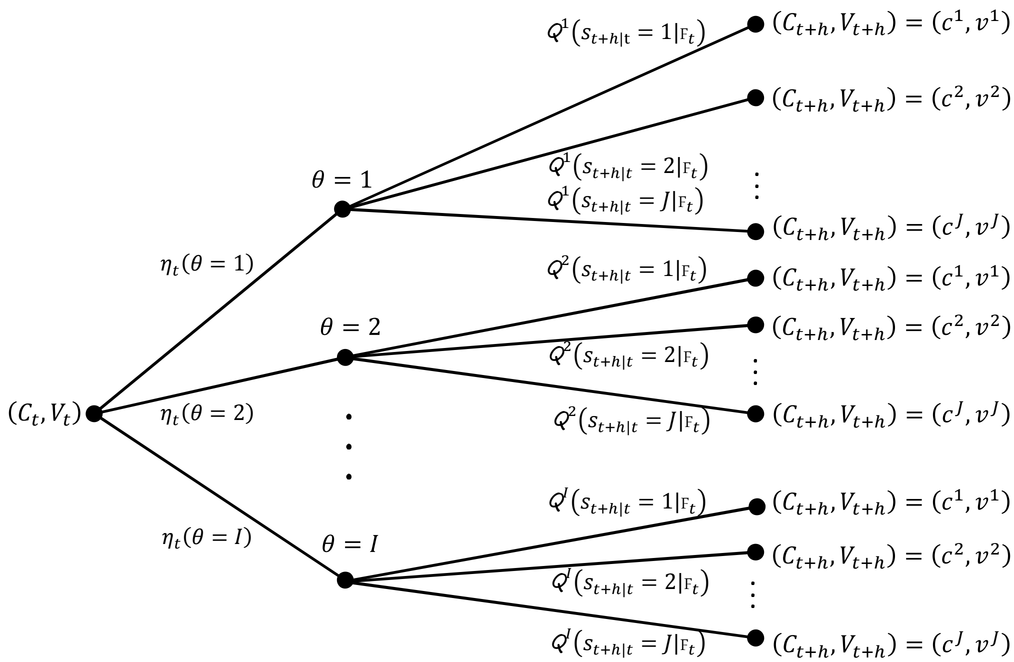

Figure 1: An illustration of the dynamic structure of ambiguity under the discrete-time setting that is faced by the decision maker at time t.

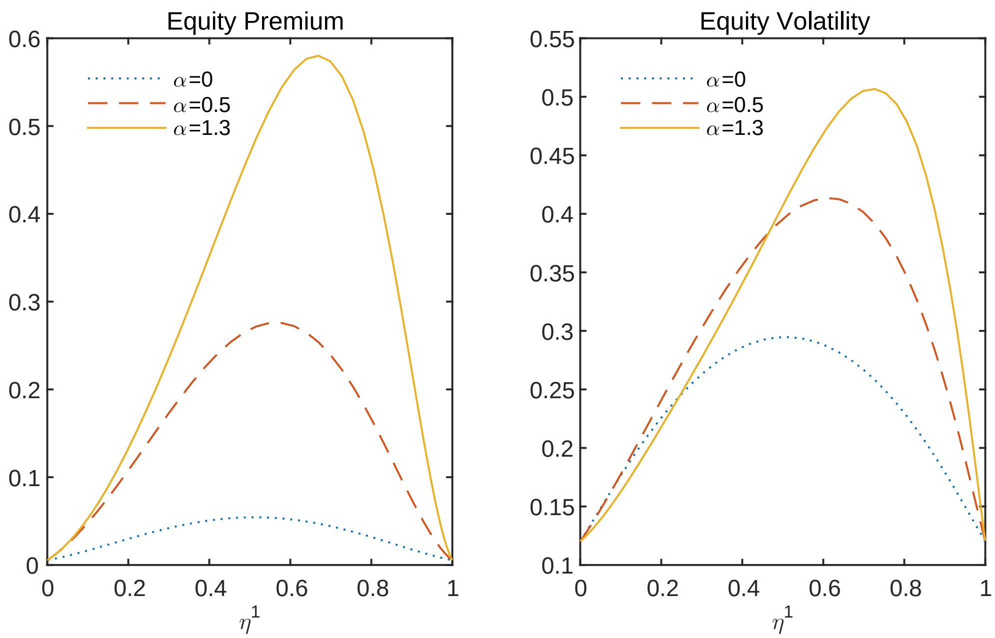

Figure 2: The equity premium (left panel) and equity volatility (right panel) conditional on η^1. The solid, broken, and dotted lines in each panel represent the corresponding conditional moments under $\alpha = 1.3$, $\alpha = 0.5$, and $\alpha = 0$, respectively. Other preference parameters are fixed at $(\beta, \rho, \gamma) = (0.023, 1/1.5, 2)$.