Continuous-Time Asset Pricing Model with Smooth Ambiguity Preferences

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1. Introduction

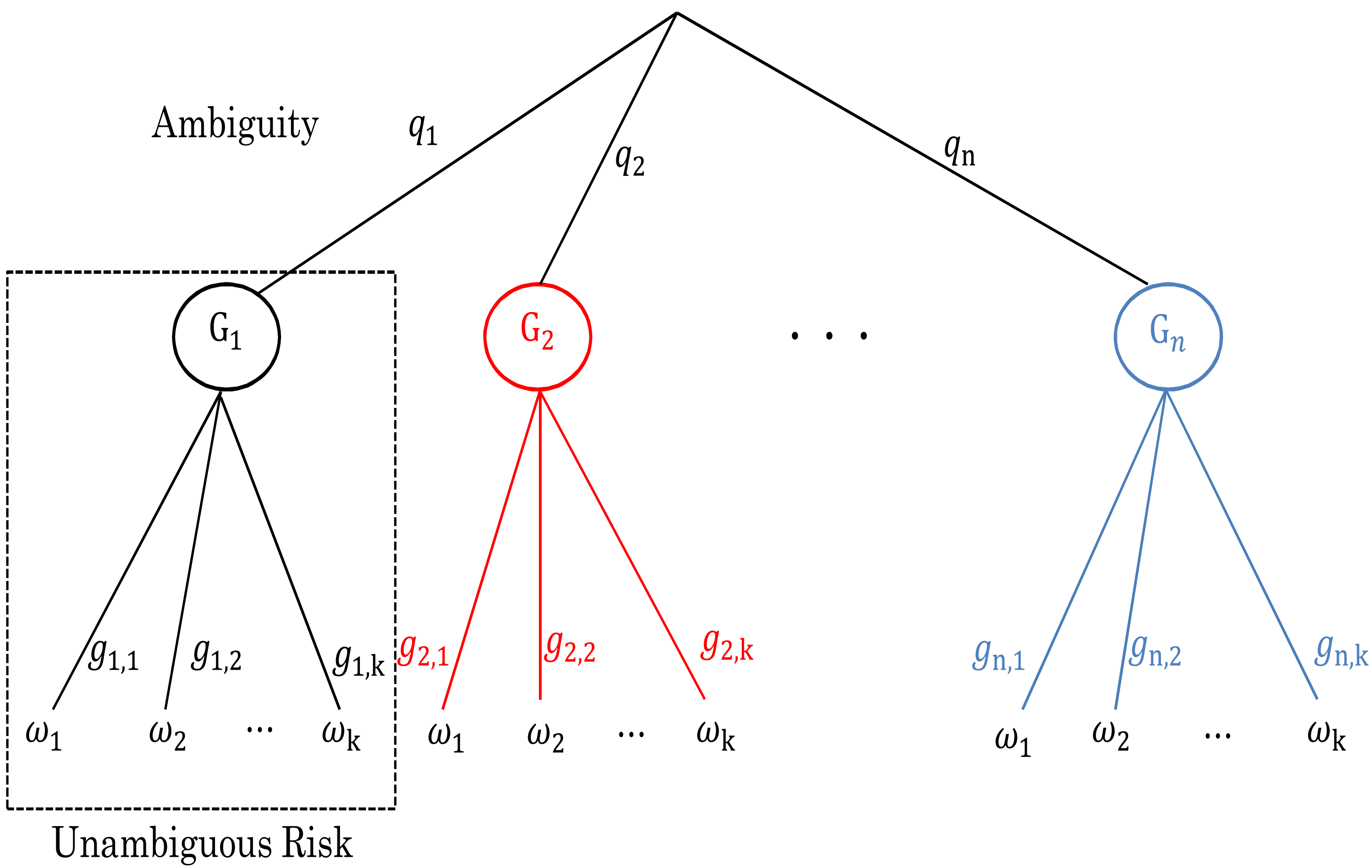
Risk & Ambiguity

The term “uncertainty” includes risk and ambiguity.

- **Risk**:
  Uncertainty of relevant payoffs whose probability measure is given.
- **Ambiguity**:
  Uncertainty of probability measures.

Most decision makers prefer a risky bet over an ambiguous one. $\Rightarrow$ Ellsberg (1961)

Example (Multiple Priors)



Ambiguity Preferences

- Subjective Expected Utility: Savege (1954)

$$V(C) = E\left[u\left(C\left(\omega\right)\right)\right] = \sum_{i,j} q_i g_{i,j}\left(\omega_j\right) u\left(C\left(\omega_j\right)\right).$$

- Maxmin Utility: Gilboa and Schmeidler (1989)

$$V(C) = \min_i E^{G_i}\left[u\left(C\left(\omega\right)\right)\right].$$

- Smooth Ambiguity Preferences: Klibanoff et al.(2005)

$$V(C) = \sum_i q_i \phi\left(E^{G_i}\left[u\left(C\left(\omega\right)\right)\right]\right).$$

- Skiadas (2013) shows that smooth ambiguity preferences converge to the SEU in the continuous-time limit.

Aim & Result

- Given powerful tools in the continuous-time financial modeling, it would be of great use if we can combine those with the smooth ambiguity preferences.
- We apply Yaari's (1987) "dual theory" to the KMM model, and interchange the role of the second utility with that of the second order probability.
  $\Rightarrow$ Iwaki and Osaki (2014), Atemporal Setting.
- By using this trick, we can prevent DM's ambiguity attitude from disappearing in the continuous-time limit.
- DM's preferences are eventually represented by the SEU with distorted beliefs.
- Our model replicates Ju and Miao's (2012) asset pricing results.

Related Literature

	Maxmin Utility	Smooth Ambiguity Preferences
Atemporal Model	Gilboa and Schmeidler (1989)	Klibanoff et al. (2005)
Discrete-Time Model	Epstein and Schneider (2003)	Klibanoff et al. (2009)
Continuous-Time Model	Chen and Epstein (2002)	This Paper

2. Discrete-Time Smooth Ambiguity Preferences

Recursive Utility without Ambiguity

- DM's time-t utility of the continuation consumption stream, $\{C_s\}_{s=t}^{T}$, is denoted by V_t.
- This paper considers the following recursive utility proposed by Epstein and Zin (1989):

$$V_t = W\left(C_t, m_t\left(V_{t+h}\right)\right). \tag{1}$$

- $m_t\left(V_{t+h}\right)$ is a time-t certainty equivalent for the uncertain next period utility, V_{t+h}.
- Supposedly, Kreps and Porteus's (1978) CE would be the most commonly used in economics and finance studies.

$$m_t^{\mathrm{KP}}\left(V_{t+h}\right) = u^{-1}\left(E_t\left[u\left(V_{t+h}\right)\right]\right),$$

where $u(\cdot)$ is a von Nuemann-Morgenstern utility index.

- It is assumed that $V_t(\omega) \in (0, \infty)$ for $\forall (t, \omega)$.

KMM Model

- There are multiple first order probability measures, $\{\mathbb{Q}^\theta\}_{\theta\in\Theta}$, where Θ is a parameter space.
- One of $\mathbb{Q}^\theta$ is selected as DM's reference probability measure, $\mathbb{P}$.
- DM attaches second order probability, $\eta_t(\theta)$, for each $\theta \in \Theta$.
- In KMM model, DM's time-t CE has the following form:

$$m_t^{\mathrm{KMM}}(V_{t+h}) = v^{-1}\left(\int_\Theta v\left(m_t^\theta(V_{t+h})\right)\eta_t(\theta)\,d\theta\right),$$

  where m_t^θ represents the KP CE conditional on $\mathbb{Q}^\theta$.
- The concavity of $v(\cdot)$ captures DM's ambiguity aversion.

Proposition 1 (Skiadas (2013)) *Under the discrete-time approximation of Brownian uncertainty, the KMM CE is expressed as follows:*

$$\begin{aligned} m_t^{\text{KMM}}(V_{t+h}) &= u^{-1}\left(E_t^{\overline{\theta}}\left[u\left(V_{t+h}\right)\right]\right) + o(h) \\ &= m_t^{\overline{\theta}}(V_{t+h}) + o(h), \end{aligned}$$

where $E_t^{\overline{\theta}}[\cdot]$ and $m_t^{\overline{\theta}}(\cdot)$ are, respectively, the time-t conditional expectation and the KP CE under the compound probability measure:

$$\mathbb{Q}^{\overline{\theta}}(\omega) \equiv \int_{\Theta} \mathbb{Q}^{\theta}(\omega)\eta(\theta)\,d\theta.$$

3. Dual Theory of Smooth Ambiguity Preferences

Yaari's Dual Theory without Ambiguity

- Atemporal setting with risky consumption, $C(\omega)$.
- Define the decumulative distribution function of $C(\omega)$ as:

$$G_C(z) \equiv \int_\Omega \mathbb{I}\left\{C(\omega) > z\right\} d\mathbb{P}(\omega), \quad \text{for } z \in (0, \infty),$$

- Also define a continuous, strictly increasing function $\varphi(\cdot) : [0,1] \to [0,1]$.
- Then, in Yaari's dual theory, DM makes his/her decision through:

$$\int_0^\infty \varphi\left[G_C(z)\right] dz = -\int_0^\infty z d\varphi\left[G_C(z)\right] = \int_\Omega C(\omega) d\tilde{\mathbb{P}}(\omega),$$

  where $\tilde{\mathbb{P}}$ is a probability measure implied by $\varphi\left[G_C(z)\right]$.
- DM is risk-averse when $\varphi(\cdot)$ is convex.

Dual Theory for KMM Model

- Iwaki and Osaki (2014) axiomatize the dual theory of KMM model under ambiguity.
- We apply their technique to the dynamic setting.
- The decumulative distribution function of $m_t^\theta(V_{t+h})$ is:

$$G_t(z) \equiv \int_\Theta \mathbb{I}\left\{m_t^\theta(V_{t+h}) > z\right\}\eta_t(\theta)\,d\theta, \quad \text{for } z \in (0,\infty).$$

- We then propose the following CE:

$$\begin{aligned} m_t^{\mathrm{D}}(V_{t+h}) &= \int_0^\infty \varphi\left[G_t(z)\right]dz \\ &= \int_\Theta m_t^\theta(V_{t+h})\,\tilde{\eta}_t(\theta)\,d\theta, \end{aligned} \tag{4}$$

where $\tilde{\eta}$ is the second order probability implied by $\varphi\left[G_t(z)\right]$.

4. Continuous-Time Smooth Ambiguity Preferences

Differential Form of CE

- From $V_t = W\left(C_t, m_t^{\mathrm{D}}\left(V_{t+h}\right)\right)$:

$$m_t^{\mathrm{D}}\left(V_{t+h}\right) = H\left(C_t, V_t, h\right),$$

for some function $H : (R_+, R, (0,1)) \to R_+$.

- Then:

$$\left.\frac{d}{dh} m_t^{\mathrm{D}}\left(V_{t+h}\right)\right|_{h=0} = \frac{\partial H}{\partial h}\left(C_t, V_t, 0\right) = -f\left(C_t, V_t\right), \quad (5)$$

where $f(c,v) \equiv -\partial H(c,v,0)/\partial h$ is determined by the functional form of $W(\cdot,\cdot)$.

Stochastic Differential Utility

- Consider Brownian uncertainty with k-dimensional BM, $B_t = \left(B_t^1, B_t^2, \cdots, B_t^k\right)^{\mathsf{T}}$, defined on $(\Omega, \mathscr{F}, \mathbb{P})$.
- Each $\mathbb{Q}^\theta$ is constructed through the Radon-Nikodým derivative process:

$$\left.\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \xi_t^\theta = \exp\left\{-\frac{1}{2}\int_0^t |\theta_s|^2 \, ds + \int_0^t \theta_s^{\mathsf{T}} dB_s\right\}.$$

- $\mathbb{Q}^{\tilde{\theta}}$ is constructed through $\xi_t^{\tilde{\theta}}$, where $\tilde{\theta}_t \equiv \int_\Theta \theta_t \tilde{\eta}_t(\theta) \, d\theta$.

Proposition 2 *By assuming $V_T = 0$, $\left(f, m_t^{\mathrm{D}}\right)$ generates the following utility process:*

$$V_t = E_t^{\tilde{\theta}}\left[\int_t^T f(C_s, V_s) \, ds\right].$$

5. Application to Asset Pricing

Basic Setting

- Continuous-time analogue of Ju and Miao (2012).
- Endowment economy with an infinitely lived agent.
- Under $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{Q}^{\theta^*})$, the fundamentals (consumption and equity dividend) process is:

$$\frac{dY_t}{Y_t} \equiv \left(\frac{dC_t}{C_t}, \frac{dD_t}{D_t}\right)^\top = (\delta + \kappa\theta_t^*)\, dt + \kappa dB_t^{\theta^*},$$

$B_t^{\theta^*}$: two-dimensional standard BM
θ_t^* : two-dimensional $\mathscr{F}_t$ adopted process.

- $\theta_t^* \in \{\theta^1, \theta^2\}$ follows continuous-time Markov chain with the infinitesimal generator matrix:

$$\Lambda = \begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{pmatrix}, \quad \lambda_{12}, \lambda_{21} > 0.$$

- We assume $\kappa_C^\top \theta^1 > \kappa_C^\top \theta^2$.

Continued

- The agent cannot observe θ_t^*.
- By observing realized value of fundamentals, the agent updates $\eta_t^i \equiv \eta_t(\theta^i)$ by Bayesian.
- Agent's aggregator:

$$f(c,v) = \left(\frac{\beta}{1-\rho}\right) \frac{c^{1-\rho} - \{(1-\gamma)v\}^{\frac{1-\rho}{1-\gamma}}}{\{(1-\gamma)v\}^{\frac{\gamma-\rho}{1-\gamma}}},$$

- Transform function:

$$\varphi(g) = g^{\alpha+1},$$

which implies the following distorted second-order probabilities:

$$\tilde{\eta}_t^1 = \left(\eta_t^1\right)^{\alpha+1}, \qquad \tilde{\eta}_t^2 = 1 - \tilde{\eta}_t^1.$$

Proposition 6 *The equilibrium risk-free rate, r_t, is given by:*

$$r_t = \beta + \rho\left\{\delta_C + \kappa_C^{\top}\overline{\theta}_t\right\} - \frac{1}{2}\gamma(\gamma+1)\,|\kappa_C|^2 + (\gamma-\rho)\,L\left(\eta_t^1\right) - \rho\,\kappa_C^{\top}\left\{\overline{\theta}_t - \tilde{\theta}_t\right\},$$

where $\overline{\theta_t} \equiv \eta_t^1\theta^1 + \eta_t^2\theta^2$ *and* $\tilde{\theta}_t \equiv \tilde{\eta}_t^1\theta^1 + \tilde{\eta}_t^2\theta^2$.

Meanwhile, the equilibrium equity price is given by:

$$S_t = F\left(\eta_t^1\right) D_t, \tag{26}$$

where $F(\cdot)$ *is a solution of the ODE (27):*

Calibration Parameter

δ_C	δ_D	$\kappa_C^\top$	$\kappa_D^\top$
0.024	0.038	(0.031, 0)	(0.086, 0.084)
$(\theta^1)^\top$	$(\theta^2)^\top$	λ_{12}	λ_{21}
(0, 0)	(−2.935, 0.029)	0.031	0.675

- $E^{\theta^1}[d\ln C] = 2.4\%, E^{\theta^1}[d\ln D] = 3.0\%,$ $E^{\theta^2}[d\ln C] = -6.8\%, E^{\theta^2}[d\ln D] = -21.8\%.$
- $E[d\ln C] = E[d\ln D] = 1.9\%, \sigma(d\ln C) = 3.1\%,$ $\sigma(d\ln D) = 12.0\%, \rho(d\ln C, d\ln D) = 0.72.$
- $\mathbb{P}\{\theta_t = \theta^1\} = 96\%.$
- Preferences parameters: $(\beta, \rho, \gamma, \alpha) = (0.023, 1/1.5, 2, 1.3).$

Calibration Result

	$E(r)$	$\sigma(r)$	$E(R^{\text{eq}})$	$\sigma(R^{\text{eq}})$	$E\left(\tilde{\eta}^1\right)$
U.S. (1871-1993)	2.66	5.13	5.75	19.02	-
Ju and Miao (2012)	2.66	1.16	5.75	18.26	-
Our Model					
(ρ, γ, α)					
(1/1.5,2,1.3)	2.66	1.22	5.75	18.37	92.4
(1/1.5,2,0.5)	3.18	1.04	1.84	16.10	94.2
(1/1.5,2,0)	3.46	0.87	0.49	14.41	95.6
(2,2,0)	6.05	2.37	0.25	13.60	95.6
(3.5,3.5,0)	8.54	4.14	0.61	13.06	95.6
(3.5,3.5,1.3)	7.53	5.51	0.62	12.31	92.4

6. Conclusion

Concluding Remarks

- We succeed in representing DM's smooth ambiguity attitude under the continuous-time setting.
- Our model replicates asset pricing results in Ju and Miao (2012).
- Therefore, our model could be regarded as a continuous-time analogue of original KMM preferences.
- Yaari's dual theory is not a complete *dual* of the SEU theory.
- Economic consequence of this difference should be examined more carefully.
- Extend our model to incorporate Poisson process.